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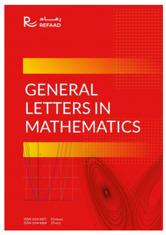

Existence Solutions For Sequential ψ-Caputo Fractional Differential Equations

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Abstract

In this manuscript, we presented the technique of having solutions to sequential ψ-Caputo fractional differential equations (ψ-CFDE) with fractional boundary conditions (ψ-FBCs). Well-known fixed point techniques are used to analyze the existence of the problem. In particular, the principle of shrinkage mapping is used to investigate the results of uniqueness. Krasnosiliki's theory reflects us in this regard obtaining the results of existence. A numerical example is employed to exemplify the desired results by considering specific cases. This demonstrates and substantiates the generalization of our work to various recent and intriguing updates.

Keywords: fractional differential equations; ψ-Caputo fractional derivative; fractional boundary conditions; existence and uniqueness.
2010 MSC: 34B10,34B15, 26A33.

1. Introduction

The idea of fractional calculus (FC) arose as an interesting research topic many years ago. The appearance of a half derivative debut by the brilliant Leibniz and L'Hospital's (see, [1]). The study of fractional-order calculus has been a subject of research for many years. It began as a result of Leibniz and L'Hospital's illustrious discourse, in which the issue of a half derivative was first raised (see, e.g., [1]). Fractional differential equations (FDEs) at present, have gained great popularity so as to have good results in applications. We refer to some of them are in polymer materials, fractional physics, theory of automatic control of abnormal diffusion, and in stochastic processes (see, [2]).

As we know that fixed-point theory (FPT) over the past 150 years, it has occupied positive progress in mathematical analysis. It has various applications in most sciences and in various fields, including the theory of improvement mathematical physics, topology, and approximation theory Poincare has launched the investigation of FPT in the nineteenth century. In 1922 the existence and uniqueness of solutions of differential and integral equations was discovered and proved by Banach for classical FPT. The existence and uniqueness of differential and integral equations solutions were established by Banach 1922 proof of a classical FPT. In 1930, Schauder came and was adopted into the Banach space of infinite dimensions,

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the first fixed point named after Schauder FPT and has many, many real-life applications in our lives. Including in game theory, economics and engineering [3, 4].

FDEs is a new approach in mathematics that is a valuable tool in modeling many phenomena in various fields such as cancer therapy, medicine, signal processing, etc.; we refer to [5, 6, 7, 8, 9, 10, 11, 12].

Recent research on ψ-Caputo fractional differential equations (ψ-CFDEs) and Caputo Hadamard fractional differential equations (CHFDEs), As we know that these two definitions are generalization of Caputo and Riemann Liouville fractional integral (RLFI) and Riemann Liouville FDs. In particular, the existence results of the solutions are investigated in [13, 14, 15, 16, 17, 18, 19, 20, 21, 22], where the strip conditions and FPT are employed.

Knowing that many authors and researchers have studied both the theory of existence and uniqueness results and stability asymptotic of higher order using Caputo and Riemann Liouville FDs,Caputo Hadamard fractional equations see [23, 24].

M. Matar et al. [25] investigated the existence and uniqueness of solutions for Hadamard fractional sequential differential equations

$$\begin{cases} \left(\mathcal{D}_{a}^{\alpha}+\gamma\mathcal{D}_{a}^{\alpha-1}\right)\varkappa(t)=f(t,\varkappa(t)),\ 1<\alpha<2, \\ \varkappa(a)=\varkappa'(a)=0, \end{cases}$$

and

$$\begin{cases} \left(\mathcal{D}_{a}^{\alpha}+\gamma\mathcal{D}_{a}^{\alpha-1}+\frac{\lambda^2-1}{4}\mathcal{D}_{a}^{\alpha-2}\right)\varkappa(t)=g(t,\varkappa(t)), \\ \varkappa(a)=\varkappa'(a)=\varkappa''(a)=0, \end{cases}$$

where $2<\alpha<3$, $t\in[a,T]$, $1\leqslant a<T$, $f,g:J\times\mathbb{R}\to\mathbb{R}$ are given continuous functions (CFs), and γ and λ are real numbers. In [26], the authors took into account the second-order infinite system of DEs

$$\begin{cases} t\frac{d^2u_i}{dt^2}+\frac{du_i}{dt}=f_j(t,u(t)),\ t\in J:=[1,q] \\ u_j(1)=u_j(q)=0, \end{cases}$$

$$\begin{cases} t\frac{d^2u_i}{dt^2}+\frac{du_i}{dt}=f_j(t,u(t)),\ t\in J:=[1,q] \\ u_j(1)=u_j(q)=0, \end{cases}$$

where $u(t)=\{u_j(t)\}_{j=1}^{\infty}$, in Banach sequence space $l^p, p\geqslant 1$.

Inspired by the above FPT and cited works, we consider ψ-CFDEs using ψ-Caputo FD boundary conditions (ψ-FBCs) of the form

$$\mathcal{D}^{\alpha_1;\psi}(\mathcal{D}^{\alpha_2;\psi}\varkappa)(\sigma)=g(\sigma,\varkappa(\sigma))\ \sigma\in J:=[\mathfrak{a},\mathfrak{b}],\ 1<\alpha_1,\alpha_2<2 \tag{1.1}$$

$$\varkappa(\mathfrak{a})=0,\ \kappa\mathcal{D}^{\vartheta_1;\psi}\varkappa(\mathfrak{b})+(1-\kappa)\mathcal{D}^{\vartheta_2;\psi}\varkappa(\mathfrak{b})=\vartheta_3,\ \vartheta_3\in\mathbb{R} \tag{1.2}$$

where $\mathcal{D}^{\alpha_1;\psi}$, $\mathcal{D}^{\alpha_2;\psi}$is the ψ-CFDEs of orders α_1, α_2, $\mathcal{D}^{\vartheta_1;\psi}$, $\mathcal{D}^{\vartheta_2;\psi}$ is the ψ-FBCs of orders ϑ_1, ϑ_2 respectively. $0<\vartheta_1,\vartheta_2<\alpha_1-\alpha_2, 0\leqslant\kappa\leqslant 1$ is some constant and a CF $g:J\times\mathbb{R}\to\mathbb{R}$.

We utilise the following hypotheses to show the results of ψ-CFDE utilise ψ-FBCs.

(Ω_1 $g:J=[\mathfrak{a},\mathfrak{b}]\times\mathbb{R}\to\mathbb{R}$ is CF.

(Ω_2) There exists nondecreasing functions $\phi_g(\sigma)\in C([\mathfrak{a},\mathfrak{b}],\mathbb{R}^+)$:

$$|g(\sigma,\varkappa)|\leqslant\phi_g(\sigma),\ \text{for any}\ \varkappa\in\mathbb{R}$$

(Ω_3) There exists the function $\Upsilon_g(\sigma)\in C([\mathfrak{a},\mathfrak{b}],\mathbb{R}^+)$:

$$|g(\sigma,\varkappa)-g(\sigma,\varkappa_1)|\leqslant\Upsilon_g(\sigma)|\varkappa-\varkappa_1|,\ \text{for any}\ \varkappa,\varkappa_1\in\mathbb{R}.$$

The famous concepts of the problem (1.1)-(1.2) and lemma are stated in [5, 6, 9, 25].

We arrange the section as follows. The following section dealt with some definitions and theories that are adopted in the rest of our work, section 3 mentions the basic results obtained, section 4 provides a numerical example of theoretical applications, and in the last section we present a conclusion.

2. Essential Preliminaries

Definition 2.1 ([1]). *For at least n-times differentiable function $\chi : [a, \infty) \to \mathbb{R}$, the Caputo's* FD *with order υ is defined by*

$$\left({}^{C}\mathfrak{D}_0^{\upsilon}\right)\chi(\sigma) = \frac{1}{\Gamma(n-\upsilon)}\int_0^{\sigma}(\sigma - s)^{n-\alpha_1-1}\chi^{(n)}(s)ds, \text{ for } n-1 < \upsilon < n,$$

where $n = [\upsilon] + 1$.

Definition 2.2 ([1]). *The RLFI of order υ for a function $\chi : [a, \infty) \to \mathbb{R}$ is defined as follows*

$$({}^{RL}\mathfrak{J}^{\upsilon})\chi(\sigma) = \frac{1}{\Gamma(\upsilon)}\int_a^{\sigma}\frac{\chi(s)}{(\sigma - s)^{1-\upsilon}}ds, \text{ for } \upsilon > 0,$$

provided the integral exists.

Definition 2.3 ([1]). *The HFI of order υ is defined by*

$$\left({}^{H}J^{\upsilon}\right)\chi(\sigma) = \frac{1}{\Gamma(\upsilon)}\int_b^{\sigma}\left(\log\frac{\sigma}{s}\right)^{\upsilon-1}\frac{\chi(s)}{s}ds, \ \upsilon > 0.$$

provided the integral exists.

Definition 2.4 ([1]). *The CHFD is defined as*

$${}^{H}\mathfrak{D}^{\upsilon}\chi(\sigma) = \frac{1}{\Gamma(n-\upsilon)}\int_a^{\sigma}\left(\log\frac{\sigma}{s}\right)^{n-\upsilon-1}\partial^n\frac{\chi(s)}{s}ds, n-1 < \upsilon < n, \ n = [\upsilon] + 1,$$

where $\chi : [a, \infty) \to \mathbb{R}$ is an n-times differentiable function and $\partial^n = \left(\sigma\frac{d}{d\sigma}\right)^n$.

Let $\psi : [\tau_1, \tau_2] \to \mathbb{R}$, be increasing via $\psi'(\sigma) \neq 0, \forall\sigma$. We start this part by defining ψ-fractional integrals and derivatives. In all notations of this section, we set

$$\partial_{\psi} = \frac{1}{\psi'(\sigma)}\frac{d}{d\sigma}.$$

Definition 2.5 ([27, 28]). The υth ψ-integral for an integrable function $\chi : [\tau_1, \tau_2] \to \mathbb{R}$ with respect to ψ is illustrated as follows :

$$\mathcal{J}_{\tau_1^+}^{\upsilon;\psi}\chi(\sigma) = \frac{1}{\Gamma(\upsilon)}\int_{\tau_1}^{\sigma}(\psi(\sigma) - \psi(\xi))^{\upsilon-1}\psi'(\xi)\chi(\xi)d\xi, \tag{2.1}$$

where

$$\Gamma(\upsilon) = \int_0^{+\infty}e^{-\sigma}\sigma^{\upsilon-1}d\sigma, \ \upsilon > 0.$$

Definition 2.6 ([27, 28]). Let $n \in \mathbb{N}$ and,$\nu \in C^n[\tau_1, \tau_2]$ be such that ψ has the same properties mentioned above. The υth ψ-fractional derivative of χ is defined by

$$\begin{aligned}\mathcal{D}_{\tau_1^+}^{\upsilon;\psi}\chi(\sigma) &= \partial_{\psi}^{(n)}\mathcal{J}_{\tau_1^+}^{n-\upsilon;\psi}\chi(\sigma)\\ &= \frac{1}{\Gamma(n-\upsilon)}\partial_{\psi}^{(n)}\int_{\tau_1}^{\sigma}(\psi(\sigma) - \psi(\xi))^{n-\upsilon-1}\psi'(\xi)\chi(\xi)d\xi,\end{aligned}$$

in which $n = [\upsilon] + 1$ .The υth ψ-Caputo derivative of χ is defined by

$${}^{C}\mathcal{D}_{\tau_1^+}^{\upsilon;\psi}\chi(\sigma) = \mathcal{J}_{\tau_1^+}^{n-\upsilon;\psi}\partial_{\psi}^{n}\chi(\sigma),$$

in which $n = [\upsilon] + 1$ for $\upsilon \notin \mathbb{N}$, $n = \upsilon$ for $\upsilon \in \mathbb{N}$. υTh ψ-Caputo derivative of χ is defined by

$${}^{C}\mathcal{D}_{\tau_1^+}^{\upsilon;\psi}\chi(\sigma) = \begin{cases}\partial_{\psi}^{n}\chi(\sigma) & \upsilon \in \mathbb{N},\\ \int_{\tau_1}^{\sigma}\frac{\psi'(\xi)(\psi(\sigma)-\psi(\xi))^{n-\upsilon-1}}{\Gamma(n-\upsilon)}\partial_{\psi}^{n}\chi(\xi)d\xi, & \upsilon \notin \mathbb{N}.\end{cases} \tag{2.2}$$

This derivative gives the Caputo-Hadamard derivative and the Caputo derivative when $\psi(\sigma) = \ln \sigma$ and $\psi(\sigma) = \sigma$, respectively. The υth ψ-Caputo derivative of the function χ is specified as ([27], Theorem 3)

$$^{C}\mathcal{D}_{\tau_1^+}^{\upsilon;\psi}\chi(\sigma) = \mathcal{D}_{\tau_1^+}^{\upsilon;\psi}\left(\chi(\sigma) - \sum_{k=0}^{n-1}\frac{\partial_{\psi}^{k}\chi(\tau_1)}{k!}(\psi(\sigma) - \psi(\tau_1))^k\right).$$

We can use the configuration rules for the above ψ-operators are recalled in this lemma.

Lemma 2.7 ([27, 28]). *Let $n-1 < \upsilon < n$ and $\chi \in C^n[\tau_1, \tau_2]$. Then the following holds:*

$$\mathcal{J}_{\tau_1^+}^{q;\psi}\left(^{C}\mathcal{D}_{\tau_1^+}^{q;\psi}\chi(\sigma)\right) = \chi(\sigma) - \sum_{k=0}^{n-1}\frac{\partial_{\psi}^{k}\chi(\tau_1)}{k!}[\psi(\sigma) - \psi(\tau_1)]^k,$$

for all $\sigma \in [\tau_1, \tau_2]$. Moreover, if $m \in \mathbb{N}$ and $\chi \in C^{n+m}[\tau_1, \tau_2]$, hence we get

$$\partial_{\psi}^{m}\left(^{C}\mathcal{D}_{\tau_1^+}^{\upsilon;\psi}\chi(\sigma)\right)(\sigma) = ^{C}\mathcal{D}_{\tau_1^+}^{\upsilon+m;\psi}\chi(\sigma) + \sum_{k=0}^{m-1}\frac{[\psi(\sigma) - \psi(\tau_1)]^{k+n-\upsilon-m}}{\Gamma(k+n-\upsilon-m+1)}\partial_{\psi}^{k+m}\chi(\tau_1).$$

Observe that if $\partial_{\psi}^{k}\chi(\tau_1) = 0, \forall k = n, n+1, ..., n+m-1$, we conclure this relation:

$$\partial_{\psi}^{m}\left(^{C}\mathcal{D}_{\tau_1^+}^{\upsilon;\psi}\chi(\sigma)\right)(\sigma) = ^{C}\mathcal{D}_{\tau_1^+}^{\upsilon+m;\psi}\chi(\sigma), \ \sigma \in [\tau_1, \tau_2]$$

Lemma 2.8 ([27, 28]). *Let $q, q' > 0$ and $\chi \in C[\tau_1, \tau_2]$. Then, $\forall \sigma \in [\tau_1, \tau_2]$ and by assuming $\mathcal{F}_{\tau_1}(\sigma) = \psi(\sigma) - \psi(\tau_1)$, we have*

1. $\mathcal{J}_{\tau_1^+}^{q;\psi}\mathcal{J}_{\tau_1^+}^{q';\psi}\chi(\sigma) = \mathcal{J}_{\tau_1^+}^{q+q';\psi}\chi(\sigma)$;
2. $^{C}\mathcal{D}_{\tau_1^+}^{q;\psi}\mathcal{J}_{\tau_1^+}^{q;\psi}\chi(\sigma) = \chi(\sigma)$;
3. $\mathcal{J}_{\tau_1^+}^{q;\psi}(\chi(\sigma))^{q'-1} = \frac{\Gamma(q')}{\Gamma(q'+q)}(\mathcal{F}_{\tau_1}(\sigma))^{q'+q-1}$;
4. $^{C}\mathcal{D}_{\tau_1^+}^{q;\psi}(\mathcal{F}_{\tau_1}(\sigma))^{q'-1} = \frac{\Gamma(q')}{\Gamma(q'-q)}(\mathcal{F}_{\tau_1}(\sigma))^{q'-q-1}$
5. $^{C}\mathcal{D}_{\tau_1^+}^{q;\psi}(\mathcal{F}_{\tau_1}(\sigma))^{k} = 0, (k = 0, ..., n-1), n \in \mathbb{N}, q \in (n-1, n]$.

Lemma 2.9 ([27, 28]). *Let $n-1 < \alpha_1 \leqslant n, \alpha_2 > 0$, $a > 0$, $\chi \in \mathcal{L}(a, \mathcal{T})$, $\mathcal{D}_{a_1^+}^{\alpha_1;\psi}\chi \in \mathcal{L}(a, \mathcal{T})$. Then the differential equation*

$$\mathcal{D}_{a}^{\alpha_1;\psi}\chi = 0$$

has the unique solution

$$\chi(\sigma) = w_0 + w_1(\psi(\sigma) - \psi(a)) + w_2(\psi(\sigma) - \psi(a))^2 + \cdots + w_{n-1}(\psi(\sigma) - \psi(a))^{n-1},$$

and

$$\mathcal{J}_{a}^{\alpha_1;\psi}\mathcal{D}_{a}^{\alpha_1;\psi}\chi(\sigma) = \chi(\sigma) + w_0 + w_1(\psi(\sigma) - \psi(a)) + w_2(\psi(\sigma) - \psi(a))^2 + \cdots + w_{n-1}(\psi(\sigma) - \psi(a))^{n-1},$$

with $w_\ell \in \mathbb{R}$, $\ell = 0, 1, \ldots, n-1$.

Furthermore,

$$\mathcal{D}_{a}^{\alpha_1;\psi}\mathcal{J}_{a}^{\alpha_1;\psi}\chi(\sigma) = \chi(\sigma),$$

and

$$\mathcal{J}_{a}^{\alpha_1;\psi}\mathcal{J}_{a}^{\alpha_2;\psi}\chi(\sigma) = \mathcal{J}_{a}^{\alpha_2;\psi}\mathcal{J}_{a}^{\alpha_1;\psi}\chi(\sigma) = \mathcal{J}_{a}^{\alpha_1+\alpha_2;\psi}\chi(\sigma).$$

Lemma 2.10. *The ψ-FBCs*

$$\mathcal{D}^{\alpha_1;\psi}(\mathcal{D}^{\alpha_2;\psi}\varkappa)(\sigma)=\varphi(\sigma),\ \sigma\in J:=[\mathfrak{a},\mathfrak{b}],\ 1<\alpha_1,\alpha_2\leqslant 2 \tag{2.3}$$

$$\kappa\mathcal{D}^{\vartheta_1;\psi}\varkappa(\mathfrak{b})+(1-\kappa)\mathcal{D}^{\vartheta_2;\psi}\varkappa(\mathfrak{b})=\vartheta_3,\varkappa(\mathfrak{a})=0 \tag{2.4}$$

is equivalent to

$$\begin{aligned}\varkappa(\sigma)=\mathcal{J}^{\alpha_2;\psi}(\mathcal{J}^{\alpha_1;\psi}\varphi)(\sigma)+\tfrac{(\psi(\sigma)-\psi(\mathfrak{a}))^{\alpha_2}}{\lambda_1\Gamma(\alpha_2+1)}(\vartheta_3-\kappa\mathcal{J}^{\alpha_2;\psi}(\mathcal{J}^{\alpha_1-\vartheta_1;\psi}\varphi)(\mathfrak{b})\\ -(1-\kappa)\mathcal{J}^{\alpha_2;\psi}(\mathcal{J}^{\alpha_1-\vartheta_2;\psi}\varphi)(\mathfrak{b})),\ \sigma\in J:=[\mathfrak{a},\mathfrak{b}],\end{aligned} \tag{2.5}$$

where

$$\lambda_1=\frac{\kappa(\psi(\mathfrak{b})-\psi(\mathfrak{a}))^{1-\vartheta_1}}{\Gamma(2-\vartheta_1)}+\frac{(1-\kappa)(\psi(\mathfrak{b})-\psi(\mathfrak{a}))^{1-\vartheta_2}}{\Gamma(2-\vartheta_2)}\neq 0. \tag{2.6}$$

Proof. Taking the υth ψ-integral from Definition 2.5 in Equation (2.3), we obtain

$$\varkappa(\sigma)=\mathcal{J}^{\alpha_2;\psi}(\mathcal{J}^{\alpha_1;\psi}\varphi)(\sigma)+c_1+c_2\frac{(\psi(\sigma)-\psi(\mathfrak{a}))^{\alpha_2}}{\Gamma(\alpha_2+1)} \tag{2.7}$$

The first boundary condition of (2.4)$\Rightarrow$ $c_1=0$ and 2nd boundary condition of (2.4), Equation (2.7), we obtain

$$\begin{aligned}\vartheta_3&=\kappa\mathcal{J}^{\alpha_2;\psi}(\mathcal{J}^{\alpha_1;\psi}\varphi)(\mathfrak{b})+c_2\kappa\frac{(\psi(\mathfrak{b})-\psi(\mathfrak{a}))^{1-\vartheta_1}}{\Gamma(2-\vartheta_1)}\\ &\quad+(1-\kappa)\mathcal{J}^{\alpha_2;\psi}(\mathcal{J}^{\alpha_1;\psi}\varphi)(\mathfrak{b})+c_2(1-\kappa)\frac{(\psi(\mathfrak{b})-\psi(\mathfrak{a}))^{1-\vartheta_2}}{\Gamma(2-\vartheta_2)}\end{aligned} \tag{2.8}$$

$$c_2=\frac{1}{\lambda_1}(\vartheta_3-\kappa\mathcal{J}^{\alpha_2;\psi}(\mathcal{J}^{\alpha_1-\vartheta_1;\psi}\varphi)(\mathfrak{b})-(1-\kappa)\mathcal{J}^{\alpha_2;\psi}(\mathcal{J}^{\alpha_1-\vartheta_2;\psi}\varphi)(\mathfrak{b})) \tag{2.9}$$

Substituting constant c_2 in (2.7), we get Equation (2.5). Finished the proof. □

Theorem 2.11. *[32] Krasnoselskii's FPT] Let a Banach space $\mathcal{X}$, select a closed, bounded, and convex set $\emptyset\neq\mathfrak{B}\subset\mathcal{X}$. Let Q_1 and Q_2 be two mappings:*

(i) $Q_1\varkappa+Q_2y\in\mathfrak{B}$ *whenever* $\varkappa,y\in\mathfrak{B}$,
(ii) Q_1 *is compact and continuous;*
(iii) Q_2 *is a contraction mapping. Moreover,* $\exists z\in\mathfrak{B}:z=Q_1z+Q_2z$.

3. Main Results

We start by defining $\zeta=C([\mathfrak{a},\mathfrak{b}],\mathbb{R}^+):[\mathfrak{a},\mathfrak{b}]\to\mathbb{R}$ as the Banach space of all CFs with the norm $\|\varkappa\|=\sup\{|\varkappa(\sigma)|,\sigma\in[\mathfrak{a},\mathfrak{b}]\}$. Now, define the mapping $\Phi:C([\mathfrak{a},\mathfrak{b}],\mathbb{R})\to C([\mathfrak{a},\mathfrak{b}],\mathbb{R})$ by

$$\begin{aligned}\Phi\varkappa(\sigma)=\mathcal{J}^{\alpha_2;\psi}\left(\mathcal{J}^{\alpha_1;\psi}(g_\varkappa)\right)(\sigma)+\tfrac{(\psi(\sigma)-\psi(\mathfrak{a}))^{\alpha_2}}{\lambda_1\Gamma(\alpha_2+1)}(\vartheta_3-\kappa\mathcal{J}^{\alpha_2;\psi}\left(\mathcal{J}^{\alpha_1-\vartheta_1;\psi}(g_\varkappa)\right)(\mathfrak{b})\\ -(1-\kappa)\mathcal{J}^{\alpha_2;\psi}(\mathcal{J}^{\alpha_1-\vartheta_2;\psi}(g_\varkappa))(\mathfrak{b})),\ \sigma\in J=[\mathfrak{a},\mathfrak{b}],\end{aligned} \tag{3.1a}$$

abbreviate $g(\sigma,\varkappa(\sigma))$ by $g_\varkappa(\sigma)$

$$\begin{aligned}\mathcal{J}^{\alpha_2;\psi}(\mathcal{J}^{\alpha_1;\psi}(g_\varkappa))(\sigma)=\frac{1}{\Gamma(\alpha_2)\Gamma(\alpha_1)}\int_{\mathfrak{a}}^{\sigma}\int_{\mathfrak{a}}^{s}(\psi(\sigma)-\psi(s))^{\alpha_2}(\psi(s)-\psi(\sigma))^{\alpha_1-1}\\ \times g(\sigma,\varkappa(\sigma))\psi'(\sigma)\,d\sigma\psi'(s)\,ds.\end{aligned}$$

We choose here FPT guarantees us many recent results , see, e.g., [29, 30, 31].

Theorem 3.1 (*Contraction Mapping Principle*)**.** *Suggest that* $(\Omega_1), (\Omega_3)$ *are holds. If* $\lambda_2 \Upsilon_g^* < 1$, *where*

$$\Upsilon_g^* = \sup\{\Upsilon_g(\sigma) : \sigma \in [\mathfrak{a}, \mathfrak{b}]\}$$

$$\lambda_2 = \mathcal{J}^{\alpha_2;\psi}(\mathcal{J}^{\alpha_1;\psi}(1))(\mathfrak{b}) + \frac{|(\psi(\sigma) - \psi(\mathfrak{a}))^{\alpha_2}|}{|\lambda_1|\Gamma(\alpha_2+1)} \times (|\vartheta_3| - |\kappa|\mathcal{J}^{\alpha_2;\psi}(\mathcal{J}^{\alpha_1-\vartheta_1;\psi}(1))(\mathfrak{b}) - (|1-\kappa|)\mathcal{J}^{\alpha_2;\psi}(\mathcal{J}^{\alpha_1-\vartheta_2;\psi}(1))(\mathfrak{b})),$$

so, the problem (1.1) and (1.2) has a unique solution on J.

Proof. Let $\mathfrak{B}_r = \{\varkappa \in C : \|\varkappa\| \leqslant r\}$ be a convex and closed bounded subset of C, where the fixed constant r satisfies

$$r \geqslant \frac{p\lambda_2}{1 - \Upsilon_g^* \lambda_2}, \tag{3.2}$$

where $p = \sup\{g(t,0) : t \in [\mathfrak{a}, \mathfrak{b}]\}$. We show that $\Phi\mathfrak{B}_r \subset \mathfrak{B}_r$ and by using the triangle inequality $|g_\varkappa| \leqslant |g_\varkappa - g_0| + |g_0|$, we have

$$|\Phi\varkappa(\sigma)| \leqslant \mathcal{J}^{\alpha_2;\psi}(\mathcal{J}^{\alpha_1;\psi}(|g_\varkappa|))(\sigma) + \frac{|(\psi(\sigma) - \psi(\mathfrak{a}))^{\alpha_2}|}{|\lambda_1|\Gamma(\alpha_2+1)}(|\vartheta_3| - |\kappa|\mathcal{J}^{\alpha_2;\psi}\left(\mathcal{J}^{\alpha_1-\vartheta_1;\psi}(|g_\varkappa|)\right)(\mathfrak{b}) - (|1-\kappa|)\mathcal{J}^{\alpha_2;\psi}(\mathcal{J}^{\alpha_1-\vartheta_2;\psi}(|g_\varkappa|))(\mathfrak{b})),$$

$$\begin{aligned}
|\Phi\varkappa(\sigma)| &\leqslant \mathcal{J}^{\alpha_2;\psi}(\mathcal{J}^{\alpha_1;\psi}(|g_\varkappa - g_0| + |g_0|))(\sigma) + \frac{|(\psi(\sigma) - \psi(\mathfrak{a}))^{\alpha_2}|}{|\lambda_1|\Gamma(\alpha_2+1)} \\
&\times (|\vartheta_3| - |\kappa|\mathcal{J}^{\alpha_2;\psi}(\mathcal{J}^{\alpha_1-\vartheta_1;\psi}(|g_\varkappa - g_0| + |g_0|))(\mathfrak{b}) \\
&- (|1-\kappa|)\mathcal{J}^{\alpha_2;\psi}(\mathcal{J}^{\alpha_1-\vartheta_2;\psi}(|g_\varkappa - g_0| + |g_0|))(\mathfrak{b})), \\
&\leqslant \mathcal{J}^{\alpha_2;\psi}(\mathcal{J}^{\alpha_1;\psi}(\Upsilon_g^* + p))(\sigma) + \frac{|(\psi(\sigma) - \psi(\mathfrak{a}))^{\alpha_2}|}{|\lambda_1|\Gamma(\alpha_2+1)} \\
&\times (|\vartheta_3| - |\kappa|\mathcal{J}^{\alpha_2;\psi}(\mathcal{J}^{\alpha_1-\vartheta_1;\psi}(\Upsilon_g^* + p))(\mathfrak{b}) \\
&- (|1-\kappa|)\mathcal{J}^{\alpha_2;\psi}(\mathcal{J}^{\alpha_1-\vartheta_2;\psi}(\Upsilon_g^* + p))(\mathfrak{b})), \\
&= \Upsilon_g^* r\lambda_2 + p\lambda_2 \\
&\leqslant r.
\end{aligned}$$

Moreover, $\Phi\mathfrak{B}_r \subset \mathfrak{B}_r$. Let $\varkappa_1, \varkappa_2 \in \mathfrak{B}_r$, we get

$$\begin{aligned}
|\Phi\varkappa_1(\sigma) - \Phi\varkappa_2(\sigma)| &\leqslant \mathcal{J}^{\alpha_2;\psi}(\mathcal{J}^{\alpha_1;\psi}(|g_{\varkappa_1} - g_{\varkappa_2}|))(\sigma) \\
&+ \frac{|(\psi(\sigma) - \psi(\mathfrak{a}))^{\alpha_2}|}{|\lambda_1|\Gamma(\alpha_2+1)}(|\vartheta_3| - |\kappa|\mathcal{J}^{\alpha_2;\psi}\left(\mathcal{J}^{\alpha_1-\vartheta_1;\psi}(|g_{\varkappa_1} - g_{\varkappa_2}|)\right)(\mathfrak{b}) \\
&- (|1-\kappa|)\mathcal{J}^{\alpha_2;\psi}(\mathcal{J}^{\alpha_1-\vartheta_2;\psi}(|g_{\varkappa_1} - g_{\varkappa_2}|))(\mathfrak{b})), \\
&\leqslant \Upsilon_g^*\|\varkappa_1 - \varkappa_2\|\mathcal{J}^{\alpha_2;\psi}(\mathcal{J}^{\alpha_1;\psi}(1))(\sigma) + \frac{|(\psi(\sigma) - \psi(\mathfrak{a}))^{\alpha_2}|}{|\lambda_1|\Gamma(\alpha_2+1)} \\
&\times (|\vartheta_3| - |\kappa|\Upsilon_g^*\|\varkappa_1 - \varkappa_2\|\mathcal{J}^{\alpha_2;\psi}(\mathcal{J}^{\alpha_1-\vartheta_1;\psi}(1))(\mathfrak{b}) \\
&- (|1-\kappa|)\Upsilon_g^*\|\varkappa_1 - \varkappa_2\|\mathcal{J}^{\alpha_2;\psi}(\mathcal{J}^{\alpha_1-\vartheta_2;\psi}(1))(\mathfrak{b})), \\
&= \Upsilon_g^*\lambda_2\|\varkappa_1 - \varkappa_2\|,
\end{aligned}$$

$\Rightarrow |\Phi\varkappa_1(\sigma) - \Phi\varkappa_2(\sigma)| \leqslant \Upsilon_g^*\lambda_2\|\varkappa_1 - \varkappa_2\|$. Since $\Upsilon_g^*\lambda_2 < 1$, hence the mapping Φ is a contraction. Now, Φ has unique FP, demonstrating that problem (1.1)-(1.2) has a unique solution on $J = [\mathfrak{a}, \mathfrak{b}]$. □

Theorem 3.2. *Suppose* (Ω_1), (Ω_2) *are satisfied. If*

$$\Upsilon_g^*[\mathcal{J}^{\alpha_2;\psi}(\mathcal{J}^{\alpha_1;\psi}(1))(\mathfrak{b})] < 1, \tag{3.3}$$

then the problem (1.1) and (1.2) has at least one solution on $[\mathfrak{a}, \mathfrak{b}]$.

Proof. Let $\mathfrak{B}_\sigma = \{\varkappa \in C([\mathfrak{a}, \mathfrak{b}], \mathbb{R}) : \|\varkappa\| \leqslant \sigma\}$ where a constant σ satisfies $\sigma \geqslant \phi_g^*\lambda_2$ and $\phi_g^* = \sup\{\phi_g(\sigma) : \sigma \in [\mathfrak{a}, \mathfrak{b}]\}$ Divide the operator Φ into the two operators Φ_1 and Φ_2 on $\mathfrak{B}_\sigma$ with

$$\Phi_1\varkappa(\sigma) = \frac{(\psi(\sigma) - \psi(\mathfrak{a}))^{\alpha_2}}{\lambda_1\Gamma(\alpha_2+1)}(\vartheta_3 - \kappa\mathcal{J}^{\alpha_2;\psi}(\mathcal{J}^{\alpha_1-\vartheta_1;\psi}(g_\varkappa))(\mathfrak{b}) - (1-\kappa)\mathcal{J}^{\alpha_2;\psi}(\mathcal{J}^{\alpha_1-\vartheta_2;\psi}(g_\varkappa))(\mathfrak{b})),$$

and

$$\Phi_2\varkappa(\sigma) = \mathcal{J}^{\alpha_2;\psi}(\mathcal{J}^{\alpha_1;\psi}(g_\varkappa))(\sigma).$$

The ball $\mathfrak{B}_\sigma$ is a bounded, closed and convex subset of the Banach space $C([\mathfrak{a}, \mathfrak{b}], \mathbb{R})$. Here, we prove that $\Phi_1\varkappa + \Phi_2 y \in \mathfrak{B}_\sigma$. Let $\varkappa, y \in \mathfrak{B}_\sigma$, then, we have

$$\begin{aligned}|\Phi_1\varkappa(\sigma) + \Phi_2 y(\sigma)| &\leqslant \frac{|(\psi(\sigma) - \psi(\mathfrak{a}))^{\alpha_2}|}{|\lambda_1|\Gamma(\alpha_2+1)}(|\vartheta_3| - |\kappa|\mathcal{J}^{\alpha_2;\psi}(\mathcal{J}^{\alpha_1-\vartheta_1;\psi}(|g_\varkappa|))(\mathfrak{b}) \\ &\quad - (|1-\kappa|)\mathcal{J}^{\alpha_2;\psi}(\mathcal{J}^{\alpha_1-\vartheta_2;\psi}(|g_\varkappa(\mathfrak{b}))) + \mathcal{J}^{\alpha_2;\psi}(\mathcal{J}^{\alpha_1;\psi}(gy))(t) \\ &\leqslant \frac{|(\psi(\sigma) - \psi(\mathfrak{a}))^{\alpha_2}|}{|\lambda_1|\Gamma(\alpha_2+1)}(|\vartheta_3| - |\kappa|\mathcal{J}^{\alpha_2;\psi}\phi_g^*\left(\mathcal{J}^{\alpha_1-\vartheta_1;\psi}(1)\right)(\mathfrak{b}) \\ &\quad - (|1-\kappa|)\mathcal{J}^{\alpha_2;\psi}\phi_g^*(\mathcal{J}^{\alpha_1-\vartheta_2;\psi}(1))(\mathfrak{b})) + \mathcal{J}^{\alpha_2;\psi}\Psi^*(\mathcal{J}^{\alpha_1;\psi}(1))(\sigma) \\ &= \phi_g^*\lambda_2 \\ &\leqslant \sigma,\end{aligned}$$

which implies that $\Phi_1\varkappa + \Phi_2 y \in \mathfrak{B}_\sigma$. Nextly, to show Φ_2 is a contraction mapping, for $\varkappa, y \in \mathfrak{B}_\sigma$, we obtain

$$\begin{aligned}\|\Phi_2\varkappa - \Phi_2 y\| &\leqslant \mathcal{J}^{\alpha_2;\psi}(\mathcal{J}^{\alpha_1;\psi}(|g_\varkappa - gy))(\mathfrak{b}) \\ &\leqslant \Upsilon_g^*\mathcal{J}^{\alpha_2;\psi}(\mathcal{J}^{\alpha_1;\psi}(1))(\mathfrak{b})\|\varkappa - y\|,\end{aligned}$$

by (Ω_3), which is a contraction by (3.3).

Next, we demonstrate Φ_1 is continuous and compact. Since g is continuous on $[\mathfrak{a}, \mathfrak{b}] \times \mathbb{R}$, we can devise Φ_1 is continuous. For $\varkappa \in \mathfrak{B}_\sigma$,

$$\|\Phi_1\varkappa\| \leqslant \phi_g^*\lambda_3,$$

where

$$\lambda_3 = \frac{|(\psi(\sigma) - \psi(\mathfrak{a}))^{\alpha_2}|}{|\lambda_1|\Gamma(\alpha_2+1)}(|\vartheta_3| - |\kappa|\mathcal{J}^{\alpha_2;\psi}(\mathcal{J}^{\alpha_1-\vartheta_1;\psi}(1))(\mathfrak{b}) - (|1-\kappa|)\mathcal{J}^{\alpha_2;\psi}(\mathcal{J}^{\alpha_1-\vartheta_2;\psi}(1))(\mathfrak{b})).$$

This indicates that $\Phi_1\mathfrak{B}_\sigma$ is uniformly bounded. Now, we demonstrate that $\Phi_1\mathfrak{B}_\sigma$ is equicontinuous. For $\sigma_1, \sigma_2 \in [\mathfrak{a}, \mathfrak{b}] : \sigma_1 < \sigma_2$ and for $\varkappa \in \mathfrak{B}_\sigma$, we have

$$\begin{aligned}&||\Phi_1\varkappa(\sigma_1) - \Phi_1\varkappa(\sigma_2)| \\ &\leqslant \frac{|(\psi(\sigma_2) - \psi(\mathfrak{a}))^{\alpha_2} - (\psi(\sigma_1) - \psi(\mathfrak{a}))^{\alpha_2}|}{|\lambda_1|\,\Gamma(\alpha_2+1)} \\ &\times (|\vartheta_3| - |\kappa|\mathcal{J}^{\alpha_2;\psi}\left(\mathcal{J}^{\alpha_1-\vartheta_1;\psi}(|g_\varkappa|)\right)(\mathfrak{b}) - (|1-\kappa|)\mathcal{J}^{\alpha_2;\psi}(\mathcal{J}^{\alpha_1-\vartheta_2;\psi}(|g_\varkappa|))(\mathfrak{b})) \\ &\leqslant \phi_g^*\lambda_3|(\psi(\sigma_2) - \psi(\mathfrak{a}))^{\alpha_2} - (\psi(\sigma_1) - \psi(\mathfrak{a}))^{\alpha_2}|.\end{aligned}$$

It is obvious that the above expression is independent of $\varkappa$ and also tends to zero as $\sigma_1 \to \sigma_2$. Therefore $\Phi_1\mathfrak{B}_\sigma$ is equicontinuous. Hence $\Phi_1\mathfrak{B}_\sigma$ is relatively compact. Now, by applying the ArzelaAscoli theorem (see, e.g., [32]), the operator Φ_1 is compact on $\mathfrak{B}_\sigma$. Thus, Φ_1 and Φ_2 satisfy the assumptions of Theorem *2.11*. By Theorem *2.11*, we confirm that the problem (1.1) and (1.2) has at least one solution on $[\mathfrak{a}, \mathfrak{b}]$. □

4. Example

We propose a numerical example to verify our results.
Consider the ψ*-CFDEs with* ψ*-FBCs, let* $\psi(\sigma) = \log \sigma$

$$\mathcal{D}^{\frac{7}{4};\psi}\left(\mathcal{D}^{\alpha_2,\psi}\varkappa\right)(\sigma) = g(\sigma, \varkappa(\sigma)), \ \sigma \in \left(\frac{1}{3}, \frac{7}{3}\right), \tag{4.1}$$

$$\varkappa\left(\frac{1}{3}\right) = 0, \ \frac{1}{7}\mathcal{D}^{\frac{7}{3},\psi}\varkappa\left(\frac{7}{3}\right) + \frac{6}{7}\mathcal{D}^{\frac{1}{3},\psi}\varkappa\left(\frac{7}{3}\right) = \frac{8}{3}, \tag{4.2}$$

where $\alpha_1 = \frac{7}{4}, \alpha_2 = \frac{4}{3}, \mathfrak{a} = \frac{1}{3}, \mathfrak{b} = \frac{7}{3}, \vartheta_1 = \frac{1}{2}, \vartheta_2 = \frac{1}{4}, \vartheta_3 = \frac{1}{4}$ *and* $\kappa = \frac{1}{8}, \lambda_1 = 1.00201235, {}^{H}J^{\frac{4}{3}}\left({}^{H}J^{\frac{7}{4}}(1)\right)\left(\frac{7}{3}\right) = 0.026579, {}^{H}J^{\frac{4}{3}}\left({}^{H}J^{1}(1)\right)\left(\frac{7}{3}\right) = 0.9021586$
${}^{H}J^{\frac{4}{3}}\left({}^{H}J^{\frac{7}{4}}(1)\right)\left(\frac{7}{3}\right) = 0.062785, \frac{(\log 5)^{\frac{4}{3}}}{\lambda_1\Gamma\left(\frac{7}{3}\right)} = 0.482569$, *and let* $g : \left(\frac{1}{3}, \frac{7}{3}\right) \times \mathbb{R} \to \mathbb{R}$ *with*

$$g(\sigma, \varkappa(\sigma)) = \frac{1}{2}\frac{\arctan \sigma}{(2\sigma + 1)}\left(\varkappa^2 + 1\right)$$

gives, $|g(\sigma, \varkappa(\sigma)) - g(\sigma, y(\sigma))| \leqslant \Upsilon_g^* |\varkappa - y|$ and $\Upsilon_g^* = \frac{1}{2}$. *Thus*, $\Upsilon_g^*\lambda_2 = 0.862487 < 1$.
Then, by Theorem 3.1, problem (4.1) and (4.2) with $g(\sigma, \varkappa(\sigma))$ *has a unique solution on* $\left(\frac{1}{3}, \frac{7}{3}\right)$.

5. Conclusions

In this manuscript, we investigated the existence and uniqueness results of ψ-CFDEs for ψ-FBCs problem (1.1)-(1.2). The existence is obtained by utilizing Krasnoselkii's Theorem, while uniqueness is achieved through the contraction mapping principle. Additionally, we provide a numerical example to illustrate the results of the previously studied theory

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