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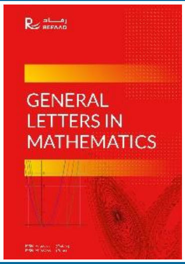

Double SEJI Integral Transform and its Applications to Differential Equations

Jinan A. Jasim [a1], Sadiq A. Mehdi [b], Emad A. Kuffi [c]

[a] Department of Mathematics, College of Education, Mustansiriyah University, Baghdad, Iraq
[b] Department of Mathematics, College of Education, Mustansiriyah University, Baghdad, Iraq
[c] Department of Mathematics, College of Basic Education, Mustansiriyah University, Baghdad, Iraq
E-mails: [a] jinanadel78@uomustansiriyah.edu.iq, [b] Sadiqmehdi71@uomustansiriyah.edu.iq, [c] emad.abbas@qu.edu.iq

Abstract

In this paper, a novel concept for a double transform in two dimensions known as the double SEJI integral transform has been proposed. Its key characteristics, including a few of its properties and theorems, have been established. A few well-known functions were also available in the Double SEJI integral transform. Later, we learn about brand-new research on partial fractional Caputo derivatives and partial differential derivatives. Finally, we apply this new transform to various first- and second-order partial differential equations.

Keywords: The double SEJI integral transform, double convolution theorem, partial derivative, partial differential equations.
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1. Introduction

Because of this, many scientific phenomena can be represented mathematically by equations built using partial differential equations [1, 2, 3, 4, 5, 6, 7, 8]. One of the most significant strategies for solving partial differential equations that has recently been discovered is the use of integral transform methods. The precise solution to the partial differential equations can then be obtained by converting differential equations into algebraic equations using integral transforms. These techniques were the result of many years of hard work by scientists and researchers, and they are today employed to resolve challenging problems in contemporary mathematics. Laplace transform, Sumudu transform, Elzaki transform, SEE complex transform, Fourier transform methods, etc. are a few examples [9, 10, 11, 12, 13, 14, 15,16, 17,18, 19,20].

Recently, more and more partial differential equations including unknown functions of two variables have been solved using numerous double integral transformations, and the solutions exceed those obtained using numerical methods. The double Sumudu transform, double Shehu transform, double Elzaki transform, new universal double integral transform, and others are additional double Laplace transform extensions [21, 22, 23, 24, 25, 26]. In order to solve fractional partial differential equations, several mathematicians have used double transformations [27, 28, 29].

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[1] Corresponding author
Email addresses: jinanadel78@uomustansiriyah.edu.iq (Jinan A. Jasim).
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In 2022, the authors introduce the SEJI integral transform, a new integral transform similar to the Laplace transform and other time-domain integral transforms $t \geq 0$.

Functions of exponential order are defined using the SEJI integral transform. We look at the following functions in the set C defined by $C = \{f(t): \exists M, L_1 \text{ and } L_2$ are greater than zero such that $|f(t)| < Me^{-iL_j|t|}$, if $t \in (-1)^j \times [0,\infty), j = 1,2\}$, where i is a complex number.

$\forall f(t) \in C$, $\exists M \in \mathbb{R}$ and L_1, L_2 may be finite or infinite. Then, SEJI integral transform denoted by $T_g^c\{f(t)\}$ is defined by the integral equation:

$$T_g^c\{f(t)\} = F_g^c(s) = p(s)\int_{t=0}^{\infty} e^{-iq(s)t} f(t)dt \qquad (1)$$

Where $t \geq 0, p(s)$ and $q(s)$ are positive real functions, i complex number. [30]

2. The Double SEJI Integral Transform

Definition 2.1: *Let $f(\chi,\gamma)$ be an integrable function defined for the variables χ and γ in the first quadrant, $p_1(s), p_2(r), q_1(s)$ and $q_2(r)$ are positive real functions; then the Double SEJI integral transform $T_{2_g}^{\ c}\{f(\chi,\gamma)\}$ is written by*

$$T_{2_g}^{\ c}\{f(\chi,\gamma)\} = F_{2_g}^{\ c}(s,r) = p_1(s)p_2(r)\int_0^{\infty}\int_0^{\infty} e^{-i(q_1(s)\chi+q_2(r)\gamma)} f(\chi,\gamma)\, d\chi\, d\gamma. \quad (2)$$

Provided that the integral exists for some $q_1(s), q_2(r)$.

The following formula is the inverse of the Double SEJI integral transform:

$$T_{2_g}^{\ c^{-1}}\left\{F_{2_g}^{\ c}(s,r)\right\} = f(\chi,\gamma) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{i}{p_1(s)} e^{iq_1(s)\chi} ds \left(\frac{1}{2\pi i}\int_{\omega-i\infty}^{\omega+i\infty} \frac{i}{p_2(r)} e^{iq_2(r)\gamma} F_{2_g}^{\ c}(s,r)\, dr\right),$$

Where γ and ω are real constants.

3. Existence Condition

If $f(\chi,\gamma)$ is an exponential order, then e and f as $\chi \to \infty, \gamma \to \infty$, and if there exist a positive constant N such that $\forall \chi \in X, \gamma \in Y$, then

$$|f(\chi,\gamma)| = Ne^{e\chi+f\gamma},$$

And we write $f(\chi,\gamma) = Oe^{e\chi+f\gamma}$, as $\chi \to \infty, \gamma \to \infty$. Or equivalently,

$$\lim_{\substack{\chi\to\infty\\ \gamma\to\infty}} e^{-i[q_1(s)\chi+q_2(r)\gamma]}|f(\chi,\gamma)| = N\lim_{\substack{\chi\to\infty\\ \gamma\to\infty}} e^{-[iq_1(s)-e]\chi-[iq_2(r)-f]\gamma} = 0,$$

$$q_1(s) > e,\ q_2(r) > f.$$

The function $f(\chi,\gamma)$ is called an exponential order as $\chi \to \infty, \gamma \to \infty$. and clearly, it does not grow faster than $Ne^{e\chi+f\gamma}$ as $\chi \to \infty, \gamma \to \infty$.

4. Properties of Double SEJI integral transform

4.1 Linearity Property

Let $T_{2_g}^{\ c}\{f(\chi,\gamma)\} = F_{2_{g_1}}^{\ c}(s,r)$ and $T_{2_g}^{\ c}\{w(x,y)\} = F_{2_{g_1}}^{\ c}(s,r)$ then for every ϱ andσ are arbitrary constants, then:

$$T_{2_g}^{\ c}\{\varrho f(\chi,\gamma) \pm \sigma w(\chi,\gamma)\} = \varrho F_{2_{g_1}}^{\ c}(s,r) \pm \sigma F_{2_{g_2}}^{\ c}(s,r).$$

Proof:

$$T_{2_g}^{\ c}\{\varrho f(\chi,\gamma) \pm \sigma w(\chi,\gamma)\} = p_1(s)p_2(r)\int_0^{\infty}\int_0^{\infty} e^{-i(q_1(s)\chi+q_2(r)\gamma)}[\varrho f(\chi,\gamma) \pm \sigma w(\chi,\gamma)]\, d\chi\, d\gamma,$$

$$= p_1(s)p_2(r)\left(\int_0^{\infty}\int_0^{\infty} e^{-i(q_1(s)\chi+q_2(r)\gamma)}\varrho f(\chi,\gamma)\, d\chi\, d\gamma \pm \int_0^{\infty}\int_0^{\infty} e^{-i(q_1(s)\chi+q_2(r)\gamma)}\sigma w(\chi,\gamma)\, d\chi\, d\gamma\right),$$

$$= p_1(s)p_2(r)\varrho\left(\int_0^{\infty}\int_0^{\infty} e^{-i(q_1(s)\chi+q_2(r)\gamma)} f(\chi,\gamma)\, d\chi\, d\gamma \pm \sigma\int_0^{\infty}\int_0^{\infty} e^{-i(q_1(s)\chi+q_2(r)\gamma)} w(\chi,\gamma)\, d\chi\, d\gamma\right),$$

$$= \varrho p_1(s)p_2(r)\left(\int_0^{\infty}\int_0^{\infty} e^{-i(q_1(s)\chi+q_2(r)\gamma)} f(\chi,\gamma)\, d\chi\, d\gamma\right) \pm \sigma p_1(s)p_2(r)\left(\int_0^{\infty}\int_0^{\infty} e^{-i(q_1(s)\chi+q_2(r)\gamma)} w(\chi,\gamma)\, d\chi\, d\gamma\right),$$

$$T_{2_g}^{c}\{\varrho f(x,y) \pm \sigma w(x,y)\} = \varrho F_{2_{g_1}}^{c}(s,r) \pm \sigma F_{2_{g_2}}^{c}(s,r). \quad \blacksquare$$

4.2 Double Convolution Theorem

In the event that $f(\chi,\gamma)$ and $h(\chi,\gamma)$ are integrable functions of two variables, then the double convolution of $f(\chi,\gamma)$ and $h(\chi,\gamma)$is given by:

$$f(\chi,\gamma) ** h(\chi,\gamma) = \int_0^{\gamma}\int_0^{\chi} f(\chi-\tau,\gamma-\mu)h(\tau,\mu)\,d\tau\,d\mu.$$

Where the double convolution with respect to χ and γ is indicated by the symbol $**$.

Theorem 4.1. *Let $F_{2_{g_1}}^{c}(s,r)$ and $F_{2_{g_2}}^{c}(s,r)$ be double SEJI integral transform of the functions $f(\chi,\gamma)$and $h(\chi,\gamma)$ respectively, we suppose that $p_1(s)p_2(r) \neq 0, \forall s,r > 0$, then*

$$T_{2_g}^{c}\{f(\chi,\gamma) ** h(\chi,\gamma)\} = \frac{1}{p_1(s)p_2(r)} F_{2_{g_1}}^{c}(s,r) F_{2_{g_2}}^{c}(s,r)$$

Proof: we have

$$T_{2_g}^{c}\{f(\chi,\gamma) ** h(\chi,\gamma)\} = p_1(s)p_2(r)\int_0^{\infty}\int_0^{\infty} e^{-i(q_1(s)\chi+q_2(r)\gamma)}\left(\int_0^{\gamma}\int_0^{\chi} f(\chi-\tau,\gamma-\mu)h(\tau,\mu)\,d\tau\,d\mu\right)d\chi\,d\gamma,$$

Substituting $\varrho = \chi-\tau, \sigma = \gamma-\mu$ and letting χ,γ to ∞, we get

$$T_{2_g}^{c}\{f(\chi,\gamma) ** h(\chi,\gamma)\} = p_1(s)p_2(r)\int_0^{\infty}\int_0^{\infty}\left(\int_{-\tau}^{\infty}\int_{-\tau}^{\infty} f(\varrho,\sigma)h(\tau,\mu)\,d\tau\,d\mu\right)e^{-i(q_1(s)(\varrho+\tau)+q_2(r)(\sigma+\mu))}\,d\varrho\,d\sigma,$$

$$= p_1(s)\int_0^{\infty}\int_0^{\infty} f(\varrho,\sigma)\left(p_2(r)\int_{-\tau}^{\infty}\int_{-\tau}^{\infty} e^{-i(q_1(s)\tau+q_2(r)\mu)}h(\tau,\mu)\,d\tau\,d\mu\right)e^{-i(q_1(s)\varrho+q_2(r)\sigma)}\,d\varrho\,d\sigma,$$

$$= \int_0^{\infty}\int_0^{\infty} f(\varrho,\sigma)\left(p_1(s)p_2(r)\int_0^{\infty}\int_0^{\infty} e^{-i(q_1(s)\tau+q_2(r)\mu)}h(\tau,\mu)\,d\tau\,d\mu\right)e^{-i(q_1(s)\varrho+q_2(r)\sigma)}\,d\varrho\,d\sigma,$$

Since $f(\chi,\gamma) \neq 0$ and $h(\chi,\gamma) \neq 0$, we have

$$T_{2_g}^{c}\{f(\chi,\gamma) ** h(\chi,\gamma)\} = T_{2_g}^{c}\{h(\chi,\gamma)\}\int_0^{\infty}\int_0^{\infty}\widetilde{w}(\tau,\mu)\,d\tau\,d\mu\int_0^{\infty}\int_0^{\infty} f(\varrho,\sigma)e^{-i(q_1(s)\varrho+q_2(r)\sigma)}\,d\varrho\,d\sigma\,,$$

$$T_{2_g}^{c}\{f(\chi,\gamma) ** h(\chi,\gamma)\} = \frac{1}{p_1(s)p_2(r)} T_{2_g}^{c}\{f(\chi,\gamma)\}T_{2_g}^{c}\{h(\chi,\gamma)\},$$

$$T_{2_g}^{c}\{f(\chi,\gamma) ** h(\chi,\gamma)\} = \frac{1}{p_1(s)p_2(r)} F_{2_{g_1}}^{c}(s,r) F_{2_{g_2}}^{c}(s,r). \quad \blacksquare$$

5. The Double SEJI Integral Transform for some Fundamental Functions:

In this section, we shall derive some elementary functions by using Double SEJI integral transform.

Formula 5.1.

$f(\chi,\gamma) = 1, \chi,\gamma > 0,$

$$T_{2_g}^{c}\{1\} = p_1(s)p_2(r)\int_0^{\infty}\int_0^{\infty} e^{-i(q_1(s)\chi+q_2(r)\gamma)}\,d\chi\,d\gamma,$$

$$= p_1(s)p_2(r)\int_0^{\infty} e^{-iq_2(r)\gamma}\left(\int_0^{\infty} e^{-iq_1(s)\chi}\,d\chi\right)d\gamma,$$

$$= p_1(s)p_2(r)\int_0^{\infty} e^{-iq_2(r)\gamma}\left(\frac{-1}{iq_1(s)}e^{-iq_1(s)\chi}\Big|_0^{\infty}\right)d\gamma = \frac{p_1(s)p_2(r)}{iq_1(s)}\left(\frac{-1}{iq_2(r)}e^{-iq_2(r)\gamma}\Big|_0^{\infty}\right),$$

$$= p_1(s)p_2(r)\int_0^{\infty} e^{-iq_2(r)y}\left(\frac{-1}{iq_1(s)}e^{-iq_1(s)\chi}\Big|_0^{\infty}\right)d\gamma = \frac{p_1(s)p_2(r)}{iq_1(s)}\left(\frac{-1}{iq_2(r)}e^{-iq_2(r)\gamma}\Big|_0^{\infty}\right),$$

$$T_{2_g}^{c}\{1\} = \frac{-p_1(s)p_2(r)}{q_1(s)q_2(r)}.$$

Formula 5.2.

$f(\chi,\gamma) = e^{\alpha\chi+\beta\gamma}, \alpha,\beta \in \mathbb{R},$

$$T_{2_g}^{c}\{e^{\alpha\chi+\beta\gamma}\} = p_1(s)p_2(r)\int_0^{\infty}\int_0^{\infty} e^{-i(q_1(s)\chi+q_2(r)\gamma)}e^{\alpha\chi+\beta\gamma}\,d\chi\,d\gamma,$$

$$= p_1(s)p_2(r)\int_0^{\infty} e^{-(iq_2(r)-\beta)\gamma}\left(\int_0^{\infty} e^{-(iq_1(s)-\alpha)\chi}\,d\chi\right)d\gamma,$$

$$= p_1(s)p_2(r)\int_0^{\infty} e^{-(iq_2(r)-\beta)\gamma}\left(\frac{-1}{iq_1(s)-\alpha}e^{e^{-(iq_1(s)-\alpha)\chi}}\Big|_0^{\infty}\right)d\gamma = \frac{p_1(s)p_2(r)}{iq_1(s)-\alpha}\left(e^{-(iq_2(r)-\beta)\gamma}\Big|_0^{\infty}\right),$$

$$= \frac{p_1(s)p_2(r)}{(iq_1(s)-\alpha)(iq_2(r)-\beta)},$$

$$T_{2_g}^{c}\{e^{\alpha\chi+\beta\gamma}\} = \frac{p_1(s)p_2(r)}{(\alpha^2+[q_1(s)]^2)(\beta^2+[q_2(r)]^2)}\big(\alpha+iq_1(s)\big)\big(\beta+iq_2(r)\big).$$

Formula 5.3.

$f(\chi,\gamma) = e^{i(\alpha\chi+\beta\gamma)}, \alpha,\beta \in \mathbb{R},$

$$T_{2_g}^{c}\{e^{i(\alpha\chi+\beta\gamma)}\} = p_1(s)p_2(r)\int_0^{\infty}\int_0^{\infty} e^{-i(q_1(s)\chi+q_2(r)\gamma)}e^{i(\alpha\chi+\beta\gamma)}\,d\chi\,d\gamma,$$

$$= p_1(s)p_2(r)\int_0^{\infty} e^{-i(q_2(r)-\beta)\gamma}\left(\int_0^{\infty} e^{-i(q_1(s)-\alpha)\chi}\,d\chi\right)d\gamma,$$

$$T_{2_g}^{c}\{e^{i(\alpha\chi+\beta\gamma)}\} = \frac{-p_1(s)p_2(r)}{(q_1(s)-\alpha)(q_2(r)-\beta)}.$$

Formula 5.4.

$f(\chi,\gamma) = \sin(\alpha\chi+\beta\gamma),$

$$T_{2_g}^{c}\{\sin(\alpha\chi+\beta\gamma)\} = p_1(s)p_2(r)\int_0^{\infty}\int_0^{\infty} e^{-i(q_1(s)\chi+q_2(r)\gamma)}\sin(\alpha\chi+\beta\gamma)\,d\chi\,d\gamma,$$

$$= p_1(s)p_2(r)\int_0^{\infty}\int_0^{\infty} e^{-i(q_1(s)\chi+q_2(r)\gamma)}\left[\frac{e^{i(\alpha\chi+\beta\gamma)}-e^{-i(\alpha\chi+\beta\gamma)}}{2i}\right]d\chi\,d\gamma,$$

$$= \frac{1}{2i}p_1(s)p_2(r)\left(\int_0^{\infty}\int_0^{\infty} e^{-i(q_1(s)\chi+q_2(r)\gamma)}e^{i(\alpha\chi+\beta\gamma)}\,d\chi\,d\gamma - \int_0^{\infty}\int_0^{\infty} e^{-i(q_1(s)\chi+q_2(r)\gamma)}e^{-i(\alpha\chi+\beta\gamma)}\,d\chi\,d\gamma\right),$$

$$= \frac{1}{2i}p_1(s)p_2(r)\left(\int_0^{\infty}\int_0^{\infty} e^{-i(q_1(s)-\alpha)\chi}e^{-i(q_2(r)-\beta)\gamma}\,d\chi\,d\gamma - \int_0^{\infty}\int_0^{\infty} e^{-i(q_1(s)+\alpha)\chi}e^{-i(q_2(r)+\beta)\gamma}\,d\chi\,d\gamma\right),$$

$$= \frac{1}{2i}\left[\frac{-p_1(s)p_2(r)}{(q_1(s)-\alpha)(q_2(r)-\beta)} + \frac{p_1(s)p_2(r)}{(q_1(s)+\alpha)(q_2(r)+\beta)}\right]$$

$$T_{2_g}^{c}\{\sin(\alpha\chi+\beta\gamma)\} = \frac{-ip_1(s)p_2(r)\big[\beta q_1(s)+\alpha q_2(r)\big]}{\Big(\big[q_1(s)\big]^2-\alpha^2\Big)\Big(\big[q_2(r)\big]^2-\beta^2\Big)}.$$

Formula 5.5.

$f(\chi,\gamma) = \cos(\alpha\chi+\beta\gamma),$

$$T_{2_g}^{c}\{\cos(\alpha\chi+\beta\gamma)\} = p_1(s)p_2(r)\int_0^{\infty}\int_0^{\infty} e^{-i(q_1(s)\chi+q_2(r)\gamma)}\cos(\alpha\chi+\beta\gamma)\,d\chi\,d\gamma,$$

$$= p_1(s)p_2(r)\int_0^{\infty}\int_0^{\infty} e^{-i(q_1(s)\chi+q_2(r)\gamma)}\left[\frac{e^{i(\alpha\chi+\beta\gamma)}+e^{-i(\alpha\chi+\beta\gamma)}}{2}\right]d\chi\,d\gamma,$$

$$= \frac{1}{2}p_1(s)p_2(r)\left(\int_0^{\infty}\int_0^{\infty} e^{-i(q_1(s)\chi+q_2(r)\gamma)}e^{i(\alpha\chi+\beta\gamma)}\,d\chi\,d\gamma + \int_0^{\infty}\int_0^{\infty} e^{-i(q_1(s)\chi+q_2(r)\gamma)}e^{-i(\alpha\chi+\beta\gamma)}\,d\chi\,d\gamma\right),$$

$$= \frac{1}{2}p_1(s)p_2(r)\left(\int_0^{\infty}\int_0^{\infty} e^{-i(q_1(s)-\alpha)\chi}e^{-i(q_2(r)-\beta)\gamma}\,d\chi\,d\gamma + \int_0^{\infty}\int_0^{\infty} e^{-i(q_1(s)+\alpha)\chi}e^{-i(q_2(r)+\beta)\gamma}\,d\chi\,d\gamma\right),$$

$$= \frac{1}{2}\left[\frac{-p_1(s)p_2(r)}{(q_1(s)-\alpha)(q_2(r)-\beta)} - \frac{p_1(s)p_2(r)}{(q_1(s)+\alpha)(q_2(r)+\beta)}\right]$$

$$T_{2_g}^{c}\{\cos(\alpha\chi+\beta\gamma)\} = \frac{-p_1(s)p_2(r)\big[q_1(s)q_2(r)+\alpha\beta\big]}{\Big(\big[q_1(s)\big]^2-\alpha^2\Big)\Big(\big[q_2(r)\big]^2-\beta^2\Big)}.$$

Formula 5.6.

$f(\chi,\gamma)=\sinh(\alpha\chi+\beta\gamma)$,

$$T_{2_g}^{c}\{\sinh(\alpha\chi+\beta\gamma)\}=p_1(s)p_2(r)\int_0^\infty\int_0^\infty e^{-i(q_1(s)\chi+q_2(r)\gamma)}\sinh(\alpha\chi+\beta\gamma)\,d\chi\,d\gamma,$$

$$=p_1(s)p_2(r)\int_0^\infty\int_0^\infty e^{-i(q_1(s)\chi+q_2(r)\gamma)}\left[\frac{e^{(\alpha\chi+\beta\gamma)}-e^{-(\alpha\chi+\beta\gamma)}}{2}\right]d\chi\,d\gamma,$$

$$=\frac{1}{2}p_1(s)p_2(r)\left(\int_0^\infty\int_0^\infty e^{-i(q_1(s)\chi+q_2(r)\gamma)}e^{(\alpha\chi+\beta\gamma)}\,d\chi\,d\gamma-\int_0^\infty\int_0^\infty e^{-i(q_1(s)\chi+q_2(r)\gamma)}e^{-(\alpha\chi+\beta\gamma)}\,d\chi\,d\gamma\right),$$

$$=\frac{1}{2}p_1(s)p_2(r)\left(\int_0^\infty\int_0^\infty e^{-(iq_1(s)-\alpha)\chi}e^{-(iq_2(r)-\beta)\gamma}\,d\chi\,d\gamma-\int_0^\infty\int_0^\infty e^{-(iq_1(s)+\alpha)\chi}e^{-(iq_2(r)+\beta)\gamma}\,d\chi\,d\gamma\right),$$

$$=\frac{1}{2}\left[\frac{-p_1(s)p_2(r)}{(iq_1(s)-\alpha)(iq_2(r)-\beta)}-\frac{p_1(s)p_2(r)}{(iq_1(s)+\alpha)(iq_2(r)+\beta)}\right]$$

$$T_{2_g}^{c}\{\sinh(\alpha\chi+\beta\gamma)\}=\frac{-ip_1(s)p_2(r)\left[\beta q_1(s)+\alpha q_2(r)\right]}{\left(\left[q_1(s)\right]^2+\alpha^2\right)\left(\left[q_2(r)\right]^2+\beta^2\right)}.$$

Formula 5.7.

$f(\chi,\gamma)=\cosh(\alpha\chi+\beta\gamma)$,

$$T_{2_g}^{c}\{\cosh(\alpha\chi+\beta\gamma)\}=p_1(s)p_2(r)\int_0^\infty\int_0^\infty e^{-i(q_1(s)\chi+q_2(r)\gamma)}\cosh(\alpha\chi+\beta\gamma)\,d\chi\,d\gamma,$$

$$=p_1(s)p_2(r)\int_0^\infty\int_0^\infty e^{-i(q_1(s)\chi+q_2(r)\gamma)}\left[\frac{e^{(\alpha\chi+\beta\gamma)}+e^{-(\alpha\chi+\beta\gamma)}}{2}\right]d\chi\,d\gamma,$$

$$=\frac{1}{2}p_1(s)p_2(r)\left(\int_0^\infty\int_0^\infty e^{-i(q_1(s)\chi+q_2(r)\gamma)}e^{(\alpha\chi+\beta\gamma)}\,d\chi\,d\gamma+\int_0^\infty\int_0^\infty e^{-i(q_1(s)\chi+q_2(r)\gamma)}e^{-(\alpha\chi+\beta\gamma)}\,d\chi\,d\gamma\right),$$

$$=\frac{1}{2}p_1(s)p_2(r)\left(\int_0^\infty\int_0^\infty e^{-(iq_1(s)-\alpha)\chi}e^{-(iq_2(r)-\beta)\gamma}\,d\chi\,d\gamma+\int_0^\infty\int_0^\infty e^{-(iq_1(s)+\alpha)\chi}e^{-(iq_2(r)+\beta)\gamma}\,d\chi\,d\gamma\right),$$

$$=\frac{1}{2}\left[\frac{-p_1(s)p_2(r)}{(iq_1(s)-\alpha)(iq_2(r)-\beta)}+\frac{p_1(s)p_2(r)}{(iq_1(s)+\alpha)(iq_2(r)+\beta)}\right]$$

$$T_{2_g}^{c}\{\cosh(\alpha\chi+\beta\gamma)\}=\frac{-p_1(s)p_2(r)\left[q_1(s)q_2(r)+\alpha\beta\right]}{\left(\left[q_1(s)\right]^2+\alpha^2\right)\left(\left[q_2(r)\right]^2+\beta^2\right)}.$$

Formula 5.8.

$f(\chi,\gamma)=(\chi\gamma)^n, n>0$

$$T_{2_g}^{c}\{(\chi\gamma)^n\}=p_1(s)p_2(r)\int_0^\infty\int_0^\infty e^{-i(q_1(s)\chi+q_2(r)\gamma)}(\chi\gamma)^n\,d\chi\,d\gamma,$$

$$=p_1(s)p_2(r)\int_0^\infty\left(\int_0^\infty e^{-iq_1(s)\chi}\;\chi^n d\chi\right)e^{-iq_2(r)\gamma}\gamma^n d\gamma$$

$$=p_1(s)p_2(r)\int_0^\infty\left[\frac{(-i)^{n+1}\Gamma(n+1)}{[q_1(s)]^{n+1}}\right]e^{-iq_2(r)\gamma}\gamma^n d\gamma$$

$$T_{2_g}^{c}\{(\chi\gamma)^n\}=\frac{(-i)^{2(n+1)}[\Gamma(n+1)]^2p_1(s)p_2(r)}{[q_1(s)q_2(r)]^{n+1}}.$$

Formula 5.9.

$f(\chi,\gamma)=\chi^m\gamma^n,\qquad m,n>0$

$$T_{2_g}^{c}\{\chi^m\gamma^n\}=p_1(s)p_2(r)\int_0^\infty\int_0^\infty e^{-i(q_1(s)\chi+q_2(r)\gamma)}(\chi^m\gamma^n)\,d\chi\,d\gamma,$$

$$=p_1(s)p_2(r)\int_0^\infty\left(\int_0^\infty e^{-iq_1(s)\chi}\;\chi^m d\chi\right)e^{-iq_2(r)\gamma}\gamma^n d\gamma,$$

$$T_{2g}^{\ c}\{\chi^m\gamma^n\} = \frac{(-i)^{m+1}(-i)^{n+1}\Gamma(m+1)\Gamma(n+1)p_1(s)p_2(r)}{[q_1(s)]^{m+1}[q_2(r)]^{n+1}}. \quad \blacksquare$$

Formula 5.10.

$f(\chi,\gamma) = h(\chi)g(\gamma).$

$$T_{2g}^{\ c}\{f(\chi,\gamma)\} = p_1(s)p_2(r)\int_0^\infty\int_0^\infty e^{-i(q_1(s)\chi+q_2(r)\gamma)}f(\chi,\gamma)\,d\chi\,d\gamma,$$

$$= \left[p_1(s)\int_0^\infty e^{-iq_1(s)\chi}h(\chi)d\chi\right]\left[p_2(r)\int_0^\infty e^{-iq_2(r)\gamma}g(\gamma)d\gamma\right],$$

$$T_{2g}^{\ c}\{f(\chi,\gamma)\} = T_g^c\{h(\chi)\}T_g^c\{g(\gamma)\}. \quad \blacksquare$$

6. The Double SEJI Integral Transform of Partial Differential Derivatives

We now provide some findings about the partial derivatives of the Double SEJI integral transform; we begin with the partial derivatives with regard to χ.

Theorem 6.1. *Let $F_{2g}^{\ c}(s,r)$ be the double SEJI integral transform of the function $f(\chi,\gamma)$, then*

i. $T_{2g}^{\ c}\left\{\frac{\partial f(x,y)}{\partial \chi}\right\} = iq_1(s)F_{2g}^{\ c}(s,r) - p_1(s)T_g^c\{f(0,\gamma)\}.$

ii. $T_{2g}^{\ c}\left\{\frac{\partial^2 f(x,y)}{\partial \chi^2}\right\} = [iq_1(s)]^2F_{2g}^{\ c}(s,r) - iq_1(s)p_1(s)T_g^c\{f(0,\gamma)\} - p_1(s)T_g^c\left\{\frac{\partial f}{\partial \chi}(0,\gamma)\right\}$

Proof: i.

$$T_{2g}^{\ c}\left\{\frac{\partial f(\chi,\gamma)}{\partial \chi}\right\} = p_1(s)p_2(r)\int_0^\infty\left(\int_0^\infty e^{-iq_1(s)\chi}\frac{\partial f(\chi,\gamma)}{\partial \chi}d\chi\right)e^{-iq_2(r)\gamma}d\gamma,$$

Integrate above by parts, we get:

$$T_{2g}^{\ c}\left\{\frac{\partial f(\chi,\gamma)}{\partial \chi}\right\} = p_1(s)p_2(r)\int_0^\infty\left(e^{-iq_1(s)\chi}\int_0^\infty\frac{\partial f(\chi,\gamma)}{\partial \chi}d\chi - \int_0^\infty\left(-iq_1(s)e^{-iq_1(s)\chi}\int_0^\infty\frac{\partial f(\chi,\gamma)}{\partial \chi}d\chi\right)d\chi\right)e^{-iq_2(r)\gamma}d\gamma,$$

$$= p_1(s)p_2(r)\int_0^\infty\left(-f(0,\gamma) + iq_1(s)\int_0^\infty e^{-iq_1(s)\chi}f(\chi,\gamma)\,d\chi\right)e^{-iq_2(r)\gamma}d\gamma,$$

$$= -p_1(s)p_2(r)\int_0^\infty -f(0,\gamma)e^{-iq_2(r)\gamma}d\gamma + iq_1(s)p_1(s)p_2(r)\int_0^\infty\int_0^\infty e^{-i(q_1(s)\chi+q_2(r)\gamma)}f(\chi,\gamma)\,d\chi\,d\gamma,$$

$$T_{2g}^{\ c}\left\{\frac{\partial f(\chi,\gamma)}{\partial \chi}\right\} = iq_1(s)F_{2g}^{\ c}(s,r) - p_1(s)T_g^c\{f(0,\gamma)\}. \quad \blacksquare$$

ii. $T_{2g}^{\ c}\left\{\frac{\partial^2 f(\chi,\gamma)}{\partial \chi^2}\right\} = p_1(s)p_2(r)\int_0^\infty\left(\int_0^\infty e^{-iq_1(s)x}\frac{\partial^2 f(\chi,\gamma)}{\partial \chi^2}d\chi\right)e^{-iq_2(r)\gamma}d\gamma,$

Integrate above by parts, we get:

$$T_{2g}^{\ c}\left\{\frac{\partial^2 f(\chi,\gamma)}{\partial \chi^2}\right\} = p_1(s)p_2(r)\int_0^\infty\left(e^{-iq_1(s)\chi}\int_0^\infty\frac{\partial f(\chi,\gamma)}{\partial \chi}d\chi - \int_0^\infty\left(-iq_1(s)e^{-iq_1(s)\chi}\int_0^\infty\frac{\partial^2 f(\chi,\gamma)}{\partial \chi^2}d\chi\right)d\chi\right)e^{-iq_2(r)\gamma}d\gamma,$$

Let $u = e^{-iq_1(s)x} \Rightarrow du = -iq_1(s)e^{-iq_1(s)x}\,dx,$

$$dv = \frac{\partial^2 f(\chi,\gamma)}{\partial \chi^2}d\chi \Rightarrow v = \int_0^\infty\frac{\partial^3 f(\chi,\gamma)}{\partial \chi^3}d\chi.$$

Then:

$$T_{2g}^{\ c}\left\{\frac{\partial^2 f(\chi,\gamma)}{\partial \chi^2}\right\} = p_2(r)\int_0^\infty e^{-iq_2(r)\gamma}d\gamma\left[p_1(s)\left(e^{-iq_1(s)\chi}\int_0^\infty\left(\frac{\partial^2 f(\chi,\gamma)}{\partial \chi^2}\right)d\chi + iq_1(s)\int_0^\infty e^{-iq_1(s)\chi}\left(\int_0^\infty\frac{\partial^2 f(\chi,\gamma)}{\partial \chi^2}d\chi\right)d\chi\right)\right],$$

$$= p_1(s)p_2(r)\int_0^\infty e^{-iq_2(r)\gamma}\left(\frac{-\partial f(0,\gamma)}{\partial x} + iq_1(s)\int_0^\infty e^{-iq_1(s)x}\frac{\partial f(\chi,\gamma)}{\partial \chi}d\chi\right)d\gamma,$$

$$= p_1(s)p_2(r)\int_0^\infty e^{-iq_2(r)\gamma}\frac{\partial^2 f(0,\gamma)}{\partial \chi^2}d\gamma + iq_1(s)p_1(s)p_2(r)\int_0^\infty\int_0^\infty e^{-i(q_1(s)\chi+q_2(r)\gamma)}\left(\frac{\partial f(\chi,\gamma)}{\partial \chi}\right)d\chi\,d\gamma,$$

$$T_{2g}^{\ c}\left\{\frac{\partial^2 f(\chi,\gamma)}{\partial \chi^2}\right\} = -p_1(s)T_g^c\left\{\frac{\partial f(0,\gamma)}{\partial \chi}\right\} + iq_1(s)T_{2g}^{\ c}\left\{\frac{\partial f(\chi,\gamma)}{\partial \chi}\right\}.$$

By substituting $T_{2g}^{\ c}\left\{\frac{\partial f(\chi,\gamma)}{\partial \chi}\right\}$ we get:

$$T_{2g}^{\ c}\left\{\frac{\partial^2 f(\chi,\gamma)}{\partial \chi^2}\right\} = [iq_1(s)]^2F_{2g}^{\ c}(s,r) - iq_1(s)p_1(s)T_g^c\{f(0,\gamma)\} - p_1(s)T_g^c\left\{\frac{\partial f}{\partial \chi}(0,\gamma)\right\}. \quad \blacksquare$$

In general,

$$T_{2_g}^{\ c}\left\{\frac{\partial^n f(\chi,\gamma)}{\partial\chi^n}\right\}=\left(iq_1(s)\right)^n F_{2_g}^{\ c}(s,r)-p_1(s)\left[\sum_{k=0}^{n-1}\left(iq_1(s)\right)^{n-k-1}T_g^c\left\{\frac{\partial^k f}{\partial\chi^k}(0,\gamma)\right\}\right].$$

We may verify the given formula. Inductively from mathematics,
for $n=1$, we proved it in the theorem
True for $n=1$.
Assume that true for $n=m$ that means:

$$T_{2_g}^{\ c}\left\{\frac{\partial^m f(\chi,\gamma)}{\partial\chi^m}\right\}=\left(iq_1(s)\right)^m F_{2_g}^{\ c}(s,r)-p_1(s)\left[\sum_{k=1}^{m}\left(iq_1(s)\right)^{m-k}T_g^c\left\{\frac{\partial^k f}{\partial\chi^k}(0,\gamma)\right\}\right],$$

We want to prove that $n=m+1$

$$T_{2_g}^{\ c}\left\{\frac{\partial^{m+1}f(\chi,\gamma)}{\partial\chi^{m+1}}\right\}=T_g^c\left\{\frac{\partial}{\partial\chi}\left(\frac{\partial^m f(\chi,\gamma)}{\partial\chi^m}\right)\right\}=\left(iq_1(s)\right)T_g^c\left\{\left(\frac{\partial^m f(\chi,\gamma)}{\partial\chi^m}\right)\right\}-p_1(s)T_g^c\left\{\frac{\partial^m f}{\partial\chi^m}(0,\gamma)\right\},$$

$$=\left(iq_1(s)\right)\left(\left(iq_1(s)\right)^m F_{2_g}^{\ c}(s,r)-p_1(s)\left[\sum_{k=1}^{m}\left(iq_1(s)\right)^{m-k}T_g^c\left\{\frac{\partial^k f}{\partial\chi^k}(0,\gamma)\right\}\right]\right)-p_1(s)T_g^c\left\{\frac{\partial^m f}{\partial\chi^m}(0,\gamma)\right\},$$

$$=\left(iq_1(s)\right)^{m+1}F_{2_g}^{\ c}(s,r)-p_1(s)\sum_{k=0}^{m-1}\left(iq_1(s)\right)^{m-k}T_g^c\left\{\frac{\partial^k f}{\partial\chi^k}(0,\gamma)\right\}-p_1(s)T_g^c\left\{\frac{\partial^m f}{\partial\chi^m}(0,\gamma)\right\},$$

$$=\left(iq_1(s)\right)^{m+1}F_{2_g}^{\ c}(s,r)-p_1(s)\left[\sum_{k=0}^{m-1}\left(iq_1(s)\right)^{m-k}T_g^c\left\{\frac{\partial^k f}{\partial\chi^k}(0,\gamma)\right\}+T_g^c\left\{\frac{\partial^m f}{\partial\chi^m}(0,\gamma)\right\}\right],$$

$$=\left(iq_1(s)\right)^{m+1}F_{2_g}^{\ c}(s,r)-p_1(s)\left[\sum_{k=0}^{m}\left(iq_1(s)\right)^{m-k-1}T_g^c\left\{\frac{\partial^k f}{\partial\chi^k}(0,\gamma)\right\}\right],$$

$$=T_{2_g}^{\ c}\left\{\frac{\partial^{m+1}f(\chi,\gamma)}{\partial\chi^{m+1}}\right\}.$$

So the theorem is true for $n\in N$. ■

Theorem 6.2. *Let $F_{2_g}^{\ c}(s,r)$ be the double SEJI integral transform of the function $f(\chi,\gamma)$, then*

i. $T_{2_g}^{\ c}\left\{\frac{\partial f(\chi,\gamma)}{\partial\gamma}\right\}=iq_2(r)F_{2_g}^{\ c}(s,r)-p_2(r)T_g^c\{f(\chi,0)\}.$

ii. $T_{2_g}^{\ c}\left\{\frac{\partial^2 f(\chi,\gamma)}{\partial\gamma^2}\right\}=\left[iq_2(r)\right]^2F_{2_g}^{\ c}(s,r)-iq_2(r)p_2(r)T_g^c\{f(\chi,0)\}-p_2(r)T_g^c\left\{\frac{\partial f}{\partial\gamma}(\chi,0)\right\}.$

iii. $T_{2_g}^{\ c}\left\{\frac{\partial^n f(\chi,\gamma)}{\partial\gamma^n}\right\}=\left(iq_2(r)\right)^n F_{2_g}^{\ c}(s,r)-p_2(r)\left[\sum_{k=0}^{n-1}\left(iq_2(r)\right)^{n-k-1}T_g^c\left\{\frac{\partial^k f}{\partial\gamma^k}(\chi,0)\right\}\right].$

This theorem has the same approach of proof but with respect to γ.

Theorem 6.3. *The mixed partial derivative's double SEJI integral transform can be calculated as follows:*

$T_{2_g}^{\ c}\left\{\frac{\partial^2 f(\chi,\gamma)}{\partial\chi\partial\gamma}\right\}=i^2q_1(s)q_2(r)F_{2_g}^{\ c}(s,r)-iq_2(r)p_1(s)T_g^c\{(0,\gamma)\}-iq_1(s)p_2(r)T_g^c\{(\chi,0)\}-p_1(s)p_2(r)f(0,0).$

Proof: we have

$$T_{2_g}^{\ c}\left\{\frac{\partial^2 f(\chi,\gamma)}{\partial\chi\partial\gamma}\right\}=p_1(s)p_2(r)\int_0^\infty\int_0^\infty e^{-i(q_1(s)\chi+q_2(r)\gamma)}\frac{\partial^2 f(\chi,\gamma)}{\partial\chi\partial y}d\chi\,d\gamma,$$

$$T_{2_g}^{\ c}\left\{\frac{\partial^2 f(\chi,\gamma)}{\partial\chi\partial\gamma}\right\}=p_1(s)\int_0^\infty e^{-iq_1(s)\chi}\left(p_2(r)\int_0^\infty e^{-iq_2(r)y}\frac{\partial^2 f(\chi,\gamma)}{\partial\chi^2}d\chi\right)dy,$$

Integrate above by parts with respect to γ, we get:

$$T_{2_g}^{\ c}\left\{\frac{\partial^2 f(\chi,\gamma)}{\partial\chi\partial\gamma}\right\}=-p_1(s)p_2(r)\int_0^\infty e^{-iq_1(s)\chi}\frac{\partial f(\chi,\gamma)}{\partial\chi}d\chi+iq_2(r)p_1(s)p_2(r)\int_0^\infty\int_0^\infty\frac{\partial f(\chi,\gamma)}{\partial\chi}e^{-i(q_1(s)\chi+q_2(r)\gamma)}d\chi\,d\gamma,$$

$$=-p_2(r)T_g^c\left\{\frac{\partial f(\chi,0)}{\partial\chi}\right\}+iq_2(r)T_{2_g}^{\ c}\left\{\frac{\partial f(\chi,\gamma)}{\partial\chi}\right\},$$

$$=-p_2(r)\left[iq_1(s)T_g^c\{f(\chi,0)\}-p_1(s)f(0,0)\right]+iq_2(r)\left[iq_1(s)T_{2_g}^{\ c}\{f(\chi,\gamma)\}-p_1(s)T_g^c\{f(0,\gamma)\}\right],$$

$$T_{2_g}^{\ c}\left\{\frac{\partial^2 f(\chi,\gamma)}{\partial\chi\partial\gamma}\right\}=i^2q_1(s)q_2(r)F_{2_g}^{\ c}(s,r)-iq_2(r)p_1(s)T_g^c\{f(0,\gamma)\}-iq_1(s)p_2(r)T_g^c\{f(\chi,0)\}-p_1(s)p_2(r)f(0,0).$$ ■

Corollary 6.1. *Let $F_{2_g}^{\ c}(s,r)$ be the double SEJI integral transform of the function $f(\chi,\gamma)$, then*

$$T_{2_g}^{c}\left\{\int_0^{\chi}\int_0^{\gamma} f(\varrho,\sigma)\,d\varrho\,d\sigma\right\} = \frac{-1}{q_1(s)q_2(r)} F_{2_g}^{c}(s,r),$$

Where $q_1(s), q_2(r) \neq 0, \ \forall s, r \in \mathbb{R}^{+}$.

Proof:

The double SEJI integral transform of the function $h(\chi,\gamma)$defined by let $T_{2_g}^{c}\{h(\chi,\gamma)\}$.

$$h(\chi,\gamma) = \int_0^{\chi}\int_0^{\gamma} f(\varrho,\sigma)\,d\varrho\,d\sigma.$$

Clearly, we have $h_{\chi\gamma}(\chi,\gamma) = f(\chi,\gamma)$ and $h(0,0) = 0$. Therefore,

$$T_{2_g}^{c}\{h_{\chi\gamma}(\chi,\gamma)\} = T_{2_g}^{c}\{f(\chi,\gamma)\} = F_{2_g}^{c}(s,r).$$

By the theorem (6.3), we obtain

$$F_{2_g}^{c}(s,r) = -q_1(s)q_2(r)T_{2_g}^{c}\{h(\chi,\gamma)\} - iq_2(r)p_1(s)T_g^{c}\{h(0,\gamma)\} - iq_1(s)p_2(r)T_g^{c}\{h(\chi,0)\} - p_1(s)p_2(r)h(0,0).$$

Thus,

$$T_{2_g}^{c}\{h(\chi,\gamma)\} = \frac{-1}{q_1(s)q_2(r)} F_{2_g}^{c}(s,r) - i\frac{p_1(s)}{q_1(s)} T_g^{c}\{h(0,\gamma)\} - \frac{ip_2(r)}{q_2(r)} T_g^{c}\{h(\chi,0)\}.$$

We have $T_g^{c}\{h(\chi,0)\} = T_g^{c}\{h(0,\gamma)\} = 0$. Then

$$T_{2_g}^{c}\{h(\chi,\gamma)\} = \frac{-1}{q_1(s)q_2(r)} F_{2_g}^{c}(s,r). \quad \blacksquare$$

Theorem 6.4. *Let* $T_{2_g}^{c}\{f(\chi,\gamma)\} = F_{2_g}^{c}(s,r)$ *, then the double SEJI integral transform for the partial fractional Caputo derivatives [31] are:*

i. $T_{2_g}^{c}\left\{\frac{\partial^{\gamma_1} f(\chi,\gamma)}{\partial\chi^{\gamma_1}}\right\} = \left(iq_1(s)\right)^{\gamma_1} F_{2_g}^{c}(s,r) - p_1(s)\left[\sum_{k=0}^{m-1}\left(iq_1(s)\right)^{\gamma_1-k-1} T_g^{c}\left\{\frac{\partial^k f}{\partial\chi^k}(0,\gamma)\right\}\right]$,

$m-1 < \gamma \leq m, \ \ m \in \mathbb{N}$

ii. $T_{2_g}^{c}\left\{\frac{\partial^{\gamma_2} f(\chi,\gamma)}{\partial\gamma^{\gamma_2}}\right\} = \left(iq_2(r)\right)^{\gamma_2} F_{2_g}^{c}(s,r) - p_2(r)\left[\sum_{k=0}^{n-1}\left(iq_2(r)\right)^{\gamma_2-k-1} T_g^{c}\left\{\frac{\partial^k f}{\partial\gamma^k}(\chi,0)\right\}\right]$,

$1 < \gamma \leq n, n \in \mathbb{N}$.

This theorem has the same approach of theorem (6.1) but with fractional orders.

The results of the double SEJI integral transform of the functions $\chi f(\chi,\gamma), \chi^2 f(\chi,\gamma), \chi^n f(\chi,\gamma), \forall n \in \mathbb{N}$ are now presented.

Theorem 6.5. *Let* $F_{2_g}^{c}(s,r)$ *be the double SEJI integral transform of the function* $f(x,y)$*, then*

i. $T_{2_g}^{c}\{\chi f(\chi,\gamma)\} = \frac{ip_1(s)}{\grave{q}_1(s)} \cdot \frac{\partial}{\partial s}\left(\frac{F_{2_g}^{c}(s,r)}{p_1(s)}\right)$.

ii. $T_{2_g}^{c}\{\chi^2 f(\chi,\gamma)\} = \frac{(i)^2 p_1(s)}{\grave{q}_1(s)} \cdot \frac{\partial^2}{\partial s^2}\left(\frac{F_{2_g}^{c}(s,r)}{p_1(s)}\right)$.

iii. $T_{2_g}^{c}\{\chi^n f(\chi,\gamma)\} = \frac{(i)^n p_1(s)}{\grave{q}_1(s)} \cdot \frac{\partial^n}{\partial s^n}\left(\frac{F_{2_g}^{c}(s,r)}{p_1(s)}\right)$.

Proof: i.

We have

$$T_{2_g}^{c}\{f(\chi,\gamma)\} = F_{2_g}^{c}(s,r) = p_1(s)p_2(r)\int_0^{\infty}\int_0^{\infty} e^{-i\left(q_1(s)\chi + q_2(r)\gamma\right)} f(\chi,\gamma)\,d\chi\,d\gamma,$$

then,

$$\frac{F_{2_g}^{c}(s,r)}{p_1(s)p_2(r)} = \int_0^{\infty}\int_0^{\infty} e^{-i(q_1(s)\chi + q_2(r)\gamma)} f(\chi,\gamma)\,d\chi\,d\gamma, \quad (3)$$

Derive both sides of eq. (3) with respect to s, we find:

$$\frac{\partial}{\partial s}\left(\frac{F_{2_g}^{c}(s,r)}{p_1(s)p_2(r)}\right) = -i\grave{q}_1(s)\int_0^{\infty}\int_0^{\infty} e^{-i(q_1(s)\chi + q_2(r)\gamma)} \chi f(\chi,\gamma)\,d\chi\,d\gamma,$$

$$= \frac{-i\grave{q}_1(s)}{p_1(s)p_2(r)} T_{2_g}^{c}\{\chi f(\chi,\gamma)\}, \quad (4)$$

Thus,

$$T_{2g}^{\ c}\{\chi f(\chi,\gamma)\} = \frac{-ip_1(s)}{\grave{q}_1(s)} \cdot \frac{\partial}{\partial s}\left(\frac{F_{2g}^{\ c}(s,r)}{p_1(s)}\right). \qquad ■$$

ii. For the proof, we drive eq. (4) with respect to s and obtain:

$$\frac{\partial}{\partial s}\left[\frac{i}{\grave{q}_1(s)}\frac{\partial}{\partial s}\left(\frac{F_{2g}^{\ c}(s,r)}{p_1(s)p_2(r)}\right)\right] = -i\grave{q}_1(s)\int_0^\infty\int_0^\infty e^{-i(q_1(s)\chi+q_2(r)\gamma)}\,\chi^2 f(\chi,\gamma)\,d\chi\,d\gamma,$$

$$= \frac{-i\grave{q}_1(s)}{p_1(s)p_2(r)}\,T_{2g}^{\ c}\{\chi^2 f(\chi,\gamma)\},$$

By simplification we get:

$$T_{2g}^{\ c}\{\chi^2 f(\chi,\gamma)\} = \frac{(i)^2 p_1(s)}{\grave{q}_1(s)} \cdot \frac{\partial^2}{\partial s^2}\left(\frac{F_{2g}^{\ c}(s,r)}{p_1(s)}\right). \qquad ■$$

iii. To proof this statement we use the mathematical induction:
When $n = 1$, it is proved in (i).
The statement is true when $n = k$,

$$T_{2g}^{\ c}\{\chi^k f(\chi,\gamma)\} = \frac{(i)^k p_1(s)}{\grave{q}_1(s)} \cdot \frac{\partial^k}{\partial s^k}\left(\frac{F_{2g}^{\ c}(s,r)}{p_1(s)}\right),$$

that means,

$$\frac{(i)^k}{\grave{q}_1(s)} \cdot \frac{\partial^k}{\partial s^k}\left(\frac{F_{2g}^{\ c}(s,r)}{p_1(s)p_2(r)}\right) = \int_0^\infty\int_0^\infty e^{-i(q_1(s)\chi+q_2(r)\gamma)}\chi^k f(\chi,\gamma)\,d\chi\,d\gamma. \qquad (5)$$

Let $n = k + 1$, the differentiation of eq. (5) will be

$$\frac{\partial}{\partial s}\left[\frac{(i)^k}{\grave{q}_1(s)} \cdot \frac{\partial^k}{\partial s^k}\left(\frac{F_{2g}^{\ c}(s,r)}{p_1(s)p_2(r)}\right)\right] = -i\grave{q}_1(s)\int_0^\infty\int_0^\infty e^{-i(q_1(s)\chi+q_2(r)\gamma)}\chi^{k+1} f(\chi,\gamma)\,d\chi\,d\gamma,$$

$$= \frac{-i\grave{q}_1(s)}{p_1(s)p_2(r)}\,T_{2g}^{\ c}\{\chi^{k+1} f(\chi,\gamma)\},$$

Then,

$$T_{2g}^{\ c}\{\chi^{k+1} f(\chi,\gamma)\} = \frac{(i)^{k+1} p_1(s)}{\grave{q}_1(s)} \cdot \frac{\partial^{k+1}}{\partial s^{k+1}}\left(\frac{F_{2g}^{\ c}(s,r)}{p_1(s)}\right). \qquad ■$$

The next theorem has the same conclusion thanks to the same method of derivation as the previous one.

Theorem 6.6. *We have*

i. $T_{2g}^{\ c}\{\gamma f(\chi,\gamma)\} = \frac{ip_2(r)}{\grave{q}_2(r)} \cdot \frac{\partial}{\partial r}\left(\frac{F_{2g}^{\ c}(s,r)}{p_2(r)}\right).$

ii. $T_{2g}^{\ c}\{\gamma^2 f(\chi,\gamma)\} = \frac{(i)^2 p_2(r)}{\grave{q}_2(r)} \cdot \frac{\partial^2}{\partial r^2}\left(\frac{F_{2g}^{\ c}(s,r)}{p_2(r)}\right).$

iii. $T_{2g}^{\ c}\{\gamma^n f(\chi,\gamma)\} = \frac{(i)^n p_2(r)}{\grave{q}_2(r)} \cdot \frac{\partial^n}{\partial r^n}\left(\frac{F_{2g}^{\ c}(s,r)}{p_2(r)}\right).$

Similar to this, we suggest a double SEJI integral transform of the function $\chi\gamma f(\chi,\gamma)$.
In the same manner, we propose double SEJI integral transform of the function

Theorem 6.7 *We have*

$$T_{2g}^{\ c}\{\chi\gamma f(\chi,\gamma)\} = \frac{\grave{p}_1(s)\grave{p}_2(r)}{p_1(s)p_2(r)\grave{q}_1(s)\grave{q}_2(r)}F_{2g}^{\ c}(s,r) - \frac{1}{\grave{q}_1(s)\grave{q}_2(r)}\left[\frac{\partial^2}{\partial s\partial r}F_{2g}^{\ c}(s,r)\right] + \frac{p_1(s)\grave{p}_2(r)}{p_2(r)\grave{q}_1(s)\grave{q}_2(r)}\frac{\partial}{\partial s}\left(\frac{F_{2g}^{\ c}(s,r)}{p_1(s)}\right) + \frac{p_2(r)\grave{p}_1(s)}{p_1(s)\grave{q}_1(s)\grave{q}_2(r)}\frac{\partial}{\partial r}\left(\frac{F_{2g}^{\ c}(s,r)}{p_2(r)}\right).$$

Proof: According to the definition (2.1), we obtain

$$\frac{\partial^2}{\partial s\partial r}F_{2g}^{\ c}(s,r) = \frac{\partial^2}{\partial s\partial r}T_{2g}^{\ c}\{f(\chi,\gamma)\} = \frac{\partial^2}{\partial s\partial r}\left[p_1(s)p_2(r)\int_0^\infty\int_0^\infty e^{-i(q_1(s)\chi+q_2(r)\gamma)}\,fe^{-i(q_1(s)\chi+q_2(r)\gamma)}\,d\chi\,d\gamma\right], \qquad (6)$$

Equation (6) changes when the Leibnitz rule is applied with respect to s and r.

$$\frac{\partial^2}{\partial s\partial r}F_{2_g}^{c}(s,r) = \frac{\partial}{\partial s}\int_0^\infty p_1(s)e^{-iq_1(s)\chi}\left[\frac{\partial}{\partial r}\int_0^\infty p_2(r)e^{-iq_2(r)\gamma}f(\chi,\gamma)d\gamma\right]d\chi,$$

We have

$$\int_0^\infty \frac{\partial}{\partial r}\left(p_2(r)e^{-iq_2(r)\gamma}f(\chi,\gamma)\right)d\gamma = \int_0^\infty \left[\grave{p}_2(r) - i\grave{q}_2(r)p_2(r)\gamma\right]e^{-iq_2(r)\gamma}f(\chi,\gamma)d\gamma,$$

Therefore

$$\frac{\partial^2}{\partial s\partial r}F_{2_g}^{c}(s,r) = \frac{\partial}{\partial s}\int_0^\infty p_1(s)e^{-iq_1(s)\chi}\left[\int_0^\infty \left[\grave{p}_2(r) - i\grave{q}_2(r)p_2(r)\gamma\right]e^{-iq_2(r)\gamma}f(\chi,\gamma)d\gamma\right]d\chi,$$

$$\frac{\partial^2}{\partial s\partial r}F_{2_g}^{c}(s,r) = \int_0^\infty \grave{p}_2(r)e^{-iq_2(r)\gamma}\left(\int_0^\infty \frac{\partial}{\partial s}p_1(s)e^{-iq_1(s)\chi}f(\chi,\gamma)d\chi\right)d\gamma$$
$$- i\grave{q}_2(r)p_2(r)e^{-iq_2(r)\gamma}\left(\int_0^\infty \frac{\partial}{\partial s}p_1(s)e^{-iq_1(s)\chi}\gamma f(\chi,\gamma)d\chi\right)d\gamma, \quad (7)$$

A simple calculation, we give that

$$\int_0^\infty \frac{\partial}{\partial s}\left(p_1(s)e^{-iq_1(s)\chi}f(\chi,\gamma)d\chi\right) = \int_0^\infty \left[\grave{p}_1(s) - i\grave{q}_1(s)p_1(s)\chi\right]e^{-iq_1(s)\chi}f(\chi,\gamma)d\chi, \quad (8)$$

$$\int_0^\infty \frac{\partial}{\partial s}\left(p_1(s)e^{-iq_1(s)\chi}\gamma\right)f(\chi,\gamma)d\gamma = \int_0^\infty \left[\grave{p}_1(s)\gamma - i\grave{q}_1(s)p_1(s)\chi\gamma\right]e^{-iq_1(s)\chi}f(\chi,\gamma)d\chi, \quad (9)$$

Substituting (15), (16) and (17), we obtain

$$\frac{\partial^2}{\partial s\partial r}F_{2_g}^{c}(s,r) = \int_0^\infty \grave{p}_2(r)e^{-iq_2(r)\gamma}\left(\int_0^\infty \left[\grave{p}_1(s) - i\grave{q}_1(s)p_1(s)\chi\right]e^{-iq_1(s)\chi}f(\chi,\gamma)d\chi\right)d\gamma$$
$$- i\int_0^\infty p_2(r)\grave{q}_2(r)e^{-iq_2(r)\gamma}\left(\int_0^\infty \left[\grave{p}_1(s)\gamma - i\grave{q}_1(s)p_1(s)\chi\gamma\right]e^{-iq_1(s)\chi}f(\chi,\gamma)d\chi\right)d\gamma,$$
$$= \grave{p}_1(s)\grave{p}_2(r)\int_0^\infty\int_0^\infty e^{-i(q_1(s)\chi+q_2(r)\gamma)}f(\chi,\gamma)\,d\chi\,d\gamma - i\grave{q}_1(s)\grave{p}_2(r)p_1(s)\int_0^\infty\int_0^\infty e^{-i(q_1(s)\chi+q_2(r)\gamma)}\chi f(\chi,\gamma)\,d\chi\,d\gamma$$
$$- i\grave{q}_2(r)\grave{p}_1(s)p_2(r)\int_0^\infty\int_0^\infty e^{-i(q_1(s)\chi+q_2(r)\gamma)}\gamma f(\chi,\gamma)\,d\chi\,d\gamma$$
$$+ (i)^2 p_1(s)p_2(r)\grave{q}_1(s)\grave{q}_2(r)\int_0^\infty\int_0^\infty e^{-i(q_1(s)\chi+q_2(r)\gamma)}\chi\gamma f(\chi,\gamma)\,d\chi\,d\gamma,$$

$$\frac{\partial^2}{\partial s\partial r}F_{2_g}^{c}(s,r) = \frac{\grave{p}_1(s)\grave{p}_2(r)}{p_1(s)p_2(r)}F_{2_g}^{c}(s,r) - i\frac{\grave{q}_1(s)\grave{p}_2(r)}{p_2(r)}T_{2_g}^{c}\{\chi f(\chi,\gamma)\} - i\frac{\grave{q}_2(r)\grave{p}_1(s)}{p_1(s)}T_{2_g}^{c}\{\gamma f(\chi,\gamma)\}$$
$$- \grave{q}_1(s)\grave{q}_2(r)T_{2_g}^{c}\{\chi\gamma f(\chi,\gamma)\},$$

$$T_{2_g}^{c}\{\chi\gamma f(\chi,\gamma)\} = \frac{\grave{p}_1(s)\grave{p}_2(r)}{p_1(s)p_2(r)\grave{q}_1(s)\grave{q}_2(r)}F_{2_g}^{c}(s,r) - \frac{1}{\grave{q}_1(s)\grave{q}_2(r)}\frac{\partial^2}{\partial s\partial r}F_{2_g}^{c}(s,r) - (i)^2\frac{p_1(s)\grave{p}_2(r)}{p_2(r)\grave{q}_1(s)\grave{q}_2(r)}\frac{\partial}{\partial s}\left(\frac{F_{2_g}^{c}(s,r)}{p_1(s)}\right)$$
$$- (i)^2\frac{\grave{p}_1(s)p_2(r)}{p_1(s)\grave{q}_1(s)\grave{q}_2(r)}\frac{\partial}{\partial r}\left(\frac{F_{2_g}^{c}(s,r)}{p_2(r)}\right),$$

From the results obtained in theorems (6.5) (i) and (6.6) (i), it follows that

$$T_{2_g}^{c}\{\chi\gamma f(\chi,\gamma)\} = \frac{\grave{p}_1(s)\grave{p}_2(r)}{p_1(s)p_2(r)\grave{q}_1(s)\grave{q}_2(r)}F_{2_g}^{c}(s,r) - \frac{1}{\grave{q}_1(s)\grave{q}_2(r)}\left[\frac{\partial^2}{\partial s\partial r}F_{2_g}^{c}(s,r)\right] + \frac{p_1(s)\grave{p}_2(r)}{p_2(r)\grave{q}_1(s)\grave{q}_2(r)}\frac{\partial}{\partial s}\left(\frac{F_{2_g}^{c}(s,r)}{p_1(s)}\right)$$
$$+ \frac{p_2(r)\grave{p}_1(s)}{p_1(s)\grave{q}_1(s)\grave{q}_2(r)}\frac{\partial}{\partial r}\left(\frac{F_{2_g}^{c}(s,r)}{p_2(r)}\right). \quad ■$$

7. Applications

The features of this transform that were previously defined are then used to solve some partial differential equations. Assuming that $\Psi(\chi,t)$ has the double SEJI integral transform represented by $T_{2_g}^{c}\{\Psi(\chi,t)\}$, where $\Psi(\chi,t)$ be an unknown function with variables $\chi,t > 0$.

7.1. First-order partial differential equations

The general linear first-order partial differential equations have the formula

$$a\Psi_{\chi}(\chi,t)+b\Psi_t(\chi,t)+c\Psi(\chi,t)=h(\chi,t), \quad (10)$$

With initial conditions

$$\Psi(\chi,0)=f(\chi), \quad \Psi(0,t)=g(t), \quad (11)$$

where $a, b, and\ c$ are constant coefficients and $h(\chi,t), f(\chi)\ and\ g(\chi)$ are continuous functions. By applying (2) on eq. (10) and using linearity property of this transform, we get

$$aT_{2g}^{\ c}\{\Psi_{\chi}(\chi,t)\}+bT_{2g}^{\ c}\{\Psi_t(\chi,t)\}+cT_{2g}^{\ c}\{\Psi(\chi,t)\}=T_{2g}^{\ c}\{h(\chi,t)\}.$$

According to results obtained in theorems for double SEJI integral transform partial derivatives, we have

$$aiq_1(s)T_{2g}^{\ c}\{\Psi(\chi,t)\}-ap_1(s)T_g^c\{\Psi(0,t)\}+biq_2(r)T_{2g}^{\ c}\{\Psi(\chi,t)\}-bp_2(r)T_{2g}^{\ c}\{\Psi(\chi,0)\}+cT_{2g}^{\ c}\{\Psi(\chi,t)\}=T_{2g}^{\ c}\{h(\chi,t)\},$$

therefore,

$$T_{2g}^{\ c}\{\Psi(\chi,t)\}=\frac{T_{2g}^{\ c}\{h(\chi,t)\}+ap_1(s)T_g^c\{\Psi(0,t)\}+bp_2(r)T_g^c\{\Psi(\chi,0)\}}{aiq_1(s)+biq_2(r)+c}, \quad (12)$$

Such that $aiq_1(s)+biq_2(r)+c\neq 0$. Similarly, taking single SEJI integral transform to (11), we obtain respectively

$$T_g^c\{\Psi(\chi,0)\}=T_g^c\{f(\chi)\}, \quad T_g^c\{\Psi(0,t)\}=T_g^c\{g(t)\}. \quad (13)$$

Substituting (13) in (12), it follows that

$$T_{2g}^{\ c}\{\Psi(\chi,t)\}=\frac{T_{2g}^{\ c}\{h(\chi,t)\}+ap_1(s)T_g^c\{g(t)\}+bp_2(r)T_g^c\{f(\chi)\}}{aiq_1(s)+biq_2(r)+c}.$$

Example 1

The general linear first-order partial differential equations

$$\Psi_{\chi}(\chi,t)=\Psi_t(\chi,t),$$

With initial conditions

$$\Psi(\chi,0)=\chi, \quad \Psi(0,t)=t,$$

Applying (2) on the differential equation, we get

$$T_{2g}^{\ c}\{\Psi_{\chi}(\chi,t)\}=T_{2g}^{\ c}\{\Psi_t(\chi,t)\}.$$

According to results obtained in theorems for double SEJI integral transform partial derivatives, we have

$$\Rightarrow iq_1(s)T_{2g}^{\ c}\{\Psi(\chi,t)\}-p_1(s)T_g^c\{\Psi(0,t)\}=iq_2(r)T_{2g}^{\ c}\{\Psi(\chi,t)\}-p_2(r)T_{2g}^{\ c}\{\Psi(\chi,0)\},$$

$$\Rightarrow iq_1(s)T_{2g}^{\ c}\{\Psi(\chi,t)\}-p_1(s)T_g^c\{t\}=iq_2(r)T_{2g}^{\ c}\{\Psi(\chi,t)\}-p_2(r)T_{2g}^{\ c}\{\chi\}.$$

Then,

$$T_{2g}^{\ c}\{\Psi(\chi,t)\}=\frac{p_1(s)\dfrac{(-i)^2p_2(r)}{[q_2(r)]^2}-p_2(r)\dfrac{(-i)^2p_1(s)}{[q_1(s)]^2}}{iq_1(s)-iq_2(r)}=\frac{(-i)^3p_1(s)p_2(r)([q_1(s)]^2-[q_2(r)]^2)}{[q_2(r)]^2[q_1(s)]^2[q_1(s)-q_2(r)]},$$

$$\Rightarrow T_{2g}^{\ c}\{\Psi(\chi,t)\}=\frac{(-i)^3p_1(s)p_2(r)[q_1(s)+q_2(r)]}{[q_2(r)]^2[q_1(s)]^2}.$$

By taking the inverse of the double SEJI integral transform for the last equation, we obtain:

$$\Psi(x,t)=\chi+t$$

Example 2:

If we set $a=b=1, c=0$ and $h(\chi,t)=-2e^{-(\chi+t)}, f(x)=e^{-x}, g(t)=e^{-t}$ in equations (10) and (11), we have

$$T_{2g}^{\ c}\{h(\chi,t)\}=T_{2g}^{\ c}\{-2e^{-(\chi+t)}\}=-2\frac{p_1(s)p_2(r)}{(1+[q_1(s)]^2)(1+[q_2(r)]^2)}\big(-1+iq_1(s)\big)\big(-1+iq_2(r)\big),$$

$$T_g^c\{f(\chi)\}=T_g^c\{e^{-\chi}\}=\frac{-p_1(s)\big(-1+iq_1(s)\big)}{(1+[q_1(s)]^2)},$$

$$T_g^c\{g(t)\}=T_g^c\{e^{-t}\}=\frac{-p_2(r)\big(-1+iq_2(r)\big)}{(1+[q_2(r)]^2)}.$$

Then,

$$T_{2g}^{\ c}\{\Psi(\chi,t)\}$$

$$=\frac{\dfrac{-2p_1(s)p_2(r)}{(1+[q_1(s)]^2)(1+[q_2(r)]^2)}\big(-1+iq_1(s)\big)\big(-1+iq_2(r)\big)+p_1(s)\dfrac{-p_2(r)\big(-1+iq_2(r)\big)}{(1+[q_2(r)]^2)}+p_2(r)\dfrac{-p_1(s)\big(-1+iq_1(s)\big)}{(1+[q_1(s)]^2)}}{iq_1(s)+iq_2(r)},$$

$$=\frac{p_1(s)p_2(r)\big(-1+iq_1(s)\big)\big(-1+iq_2(r)\big)}{(1+[q_1(s)]^2)(1+[q_2(r)]^2)}.$$

The exact solution is

$$\Psi(\chi,t)=T_{2_g}^{c\,-1}\left\{\frac{p_1(s)p_2(r)(-1+iq_1(s))(-1+iq_2(r))}{(1+[q_1(s)]^2)(1+[q_2(r)]^2)}\right\}=e^{-(\chi+t)}$$

Example3:

$$\Psi_\chi(\chi,t)+3\Psi_t(\chi,t)+2\Psi(\chi,t)=-e^{-t}, \quad (14)$$

Subject to $f(\chi)=1$, $g(t)=e^{-t}$,
then

$$T_{2_g}^{c}\{h(\chi,t)\}=T_{2_g}^{c}\{-e^{-t}\}=\frac{-p_1(s)p_2(r)}{[q_1(s)]^2(1+[q_2(r)]^2)}iq_1(s)(-1+iq_2(r)),$$

$$T_g^c\{f(\chi)\}=T_g^c\{1\}=\frac{-ip_1(s)}{q_1(s)},$$

$$T_g^c\{g(t)\}=T_g^c\{e^{-t}\}=\frac{-p_2(r)(-1+iq_2(r))}{(1+[q_2(r)]^2)}.$$

Then,

$$T_{2_g}^{c}\{\Psi(\chi,t)\}=\frac{\frac{-p_1(s)p_2(r)}{[q_1(s)]^2(1+[q_2(r)]^2)}iq_1(s)(-1+iq_2(r))+p_1(s)\frac{-p_2(r)(-1+iq_2(r))}{(1+[q_2(r)]^2)}+p_2(r)\frac{-ip_1(s)}{q_1(s)}}{iq_1(s)+3iq_2(r)+2},$$

$$=\frac{p_1(s)p_2(r)iq_1(s)(-1+iq_2(r))}{[q_1(s)]^2(1+[q_2(r)]^2)}.$$

The exact solution for eq. (14) is

$$\Psi(\chi,t)=T_{2_g}^{c\,-1}\left\{\frac{p_1(s)p_2(r)iq_1(s)(-1+iq_2(r))}{[q_1(s)]^2(1+[q_2(r)]^2)}\right\}=e^{-t}$$

7.2. Application to telegraph equation

The form of one dimensional linear hyperbolic telegraph equation is:

$$\Psi_{tt}(\chi,t)+2a\Psi_t(\chi,t)+b^2\Psi(\chi,t)=\Psi_{\chi\chi}(\chi,t)+h(\chi,t), \quad (15)$$

with initial conditions

$$\Psi(\chi,0)=f_0(\chi),\quad \Psi_t(\chi,0)=f_1(\chi), \Psi(0,t)=g_0(t),\quad \Psi_\chi(0,t)=g_1(t),$$

where $(\chi,t)\in[0,1]\times[0,T]$, $\Psi(\chi,t)$ is unknown function, a,b are constant coefficients and $h(\chi,t), f_0(\chi), f_1(\chi), g_0(t),\ g_1(t)$ are continuous functions. By applying (2) on eq. (15), we get

$$T_{2_g}^{c}\{\Psi_{tt}(\chi,t)\}+2aT_{2_g}^{c}\{\Psi_t(\chi,t)\}+b^2T_{2_g}^{c}\{\Psi(\chi,t)\}=T_{2_g}^{c}\{\Psi_{\chi\chi}(\chi,t)\}+T_{2_g}^{c}\{h(\chi,t)\},$$

So,

$$\begin{aligned}[iq_2(r)]^2T_{2_g}^{c}\{\Psi(\chi,t)\}-iq_2(r)p_2(r)T_g^c\{\Psi(\chi,0)\}-p_2(r)T_g^c\{\Psi_t(\chi,0)\}+2aiq_2(r)T_{2_g}^{c}\{\Psi(\chi,t)\}-2ap_2(r)T_{2_g}^{c}\{\Psi(\chi,0)\}\\+b^2T_{2_g}^{c}\{\Psi(\chi,t)\}\\=[iq_1(s)]^2T_{2_g}^{c}\{\Psi(\chi,t)\}-iq_1(s)p_1(s)T_g^c\{\Psi(0,t)\}-p_1(s)T_g^c\{\Psi_\chi(0,t)\}+T_{2_g}^{c}\{h(\chi,t)\},\end{aligned}$$

therefore,

$$T_{2_g}^{c}\{\Psi(\chi,t)\}=\frac{[iq_2(r)+2a]p_2(r)T_g^c\{\Psi(x,0)\}+p_2(r)T_g^c\{\Psi_t(x,0)\}+T_{2_g}^{c}\{h(x,t)\}-iq_1(s)p_1(s)T_g^c\{\Psi(0,t)\}-p_1(s)T_g^c\{\Psi_x(0,t)\}}{[iq_2(r)]^2-[iq_1(s)]^2+2aiq_2(r)+b^2}, \quad (16)$$

Such that $[iq_2(r)]^2-[iq_1(s)]^2+2aiq_2(r)+b^2\neq 0$. Similarly, taking single SEJI integral transform to the initial conditions, we obtain

$$T_g^c\{\Psi(\chi,0)\}=T_g^c\{f_0(\chi)\},\quad T_g^c\{\Psi_t(\chi,0)\}=T_g^c\{f_1(\chi)\}, \quad (17)$$

$$T_g^c\{\Psi(0,t)\}=T_g^c\{g_0(t)\},\qquad T_g^c\{\Psi_\chi(0,t)\}=T_g^c\{g_1(t)\}. \quad (18)$$

Substituting (17), (18) in (12), we get

$$T_{2_g}^{c}\{\Psi(\chi,t)\}=\frac{[iq_2(r)+2a]p_2(r)T_g^c\{f_0(x)\}+p_2(r)T_g^c\{f_1(x)\}+T_{2_g}^{c}\{h(x,t)\}-iq_1(s)p_1(s)T_g^c\{g_0(t)\}-p_1(s)T_g^c\{g_1(t)\}}{[iq_2(r)]^2-[iq_1(s)]^2+2aiq_2(r)+b^2}, \quad (19)$$

By taking the inverse of the double SEJI integral transform for the last equation, we get the exact solution of eq. (15)

Example4:

Consider the following telegraph equations

$$\Psi_{tt}(\chi,t)+2a\Psi_t(\chi,t)+b^2\Psi(\chi,t)=\Psi_{\chi\chi}(\chi,t)+(-3-4a+b^2)e^{-2t}\sinh\chi, \quad (20)$$

with initial conditions

$$\Psi(\chi,0)=f_0(x)=\sinh\chi,\quad \Psi_t(\chi,0)=f_1(\chi)=-2\sinh\chi,$$
$$\Psi(0,t)=g_0(t)=0,\quad \Psi_\chi(0,t)=g_1(t)=e^{-2t},\qquad t\geq 0$$

We have

$$T_{2g}^{\,c}\{h(\chi,t)\}=\frac{(-3-4a+b^2)}{2}\left[T_{2g}^{\,c}\{e^{\chi-2t}\}-T_{2g}^{\,c}\{e^{-\chi-2t}\}\right],$$
$$=\frac{(-3-4a+b^2)}{2}\left[\frac{p_1(s)p_2(r)\big(1+iq_1(s)\big)\big(-2+iq_2(r)\big)}{(1+[q_1(s)]^2)(4+[q_2(r)]^2)}-\frac{p_1(s)p_2(r)\big(-1+iq_1(s)\big)\big(-2+iq_2(r)\big)}{(1+[q_1(s)]^2)(4+[q_2(r)]^2)}\right],$$
$$T_{2g}^{\,c}\{h(\chi,t)\}=(-3-4a+b^2)\left[\frac{p_1(s)p_2(r)\big(-2+iq_2(r)\big)}{(1+[q_1(s)]^2)(4+[q_2(r)]^2)}\right], \quad (21)$$
$$T_g^c\{f_0(\chi)\}=T_g^c\{\sinh\chi\}=\frac{-p_1(s)}{1+[q_1(s)]^2},\quad T_g^c\{f_1(\chi)\}=T_g^c\{-2\sinh\chi\}=\frac{2p_1(s)}{1+[q_1(s)]^2}, \quad (22)$$
$$T_g^c\{g_0(t)\}=0,\quad T_g^c\{g_1(t)\}=T_g^c\{e^{-2t}\}=\frac{-p_2(r)\big(-2+iq_2(r)\big)}{4+[q_2(r)]^2}, \quad (23)$$

Then, by substituting (21), (22) and (23) in (12) with simplification, we get

$$T_{2g}^{\,c}\{\Psi(\chi,t)\}=\frac{-p_1(s)p_2(r)}{(1+[q_1(s)]^2)\big(2+iq_2(r)\big)},$$
$$T_{2g}^{\,c}\{\Psi(\chi,t)\}=\frac{-p_1(s)p_2(r)}{(1+[q_1(s)]^2)\big(2+iq_2(r)\big)}\cdot\frac{\big(2-iq_2(r)\big)}{\big(2-iq_2(r)\big)}=\frac{p_1(s)p_2(r)\big(-2+iq_2(r)\big)}{(1+[q_1(s)]^2)(4+[q_2(r)]^2)}.$$

The exact solution for eq. (20) is

$$\Psi(\chi,t)=T_{2g}^{\,c\,-1}\left\{\frac{p_1(s)p_2(r)\big(-2+iq_2(r)\big)}{(1+[q_1(s)]^2)(4+[q_2(r)]^2)}\right\}=e^{-2t}\sinh\chi.$$

Example 5:

Consider the following telegraph equations

$$\Psi_{tt}(\chi,t)+2a\Psi_t(\chi,t)+b^2\Psi(\chi,t)=\Psi_{\chi\chi}(\chi,t)-2a\sin(\chi)\sin(t)+b^2\sin(x)\cos(t), \quad (24)$$

Subject to the following initial conditions

$$\Psi(\chi,0)=f_0(\chi)=\sin(\chi),\quad \Psi_t(\chi,0)=f_1(\chi)=0,$$
$$\Psi(0,t)=g_0(t)=0,\quad \Psi_\chi(0,t)=g_1(t)=\cos(t),\qquad t\geq 0$$

By the formula 5.10, we have

$$T_{2g}^{\,c}\{h(\chi,t)\}=2aT_g^c\{\sin(\chi)\}T_g^c\{\sin(t)\}+b^2T_g^c\{\sin(\chi)\}T_g^c\{\cos(t)\},$$
$$=-2a\left[\frac{-p_1(s)}{[q_1(s)]^2-1}\right]\left[\frac{-p_2(r)}{[q_2(r)]^2-1}\right]+b^2\left[\frac{-p_1(s)}{[q_1(s)]^2-1}\right]\left[\frac{-ip_2(r)q_2(r)}{[q_2(r)]^2-1}\right],$$
$$T_{2g}^{\,c}\{h(\chi,t)\}=\frac{p_1(s)p_2(r)\big(-2a+ib^2q_2(r)\big)}{([q_1(s)]^2-1)([q_2(r)]^2-1)}, \quad (25)$$
$$T_g^c\{f_0(\chi)\}=T_g^c\{\sin(\chi)\}=\frac{-p_1(s)}{[q_1(s)]^2-1},\quad T_g^c\{f_1(\chi)\}=0, \quad (26)$$
$$T_g^c\{g_0(t)\}=0,\quad T_g^c\{g_1(t)\}=T_g^c\{\cos(t)\}=\frac{-ip_2(r)q_2(r)}{[q_2(r)]^2-1}, \quad (27)$$

Then, by substituting (25), (26) and (27) in (16) with simplification, we get

$$T_{2g}^{\,c}\{\Psi(\chi,t)\}=\frac{ip_1(s)q_2(r)}{([q_1(s)]^2-1)([q_2(r)]^2-1)},$$

The exact solution for the eq. (24) is

$$\Psi(\chi,t)=T_{2g}^{\,c\,-1}\left\{\frac{ip_1(s)q_2(r)}{([q_1(s)]^2-1)([q_2(r)]^2-1)}\right\}=\sin(x)\cos(t)$$

Klein-Gordon equation

The standard form of linear Klein-Gordon equation is:

$$\Psi_{tt}(\chi,t)-\Psi_{\chi\chi}(\chi,t)+a\Psi(\chi,t)=h(\chi,t), \quad (28)$$

with initial conditions

$$\Psi(\chi,0)=f_0(\chi),\quad \Psi_t(\chi,0)=f_1(\chi),\qquad \Psi(0,t)=g_0(t),\quad \Psi_x(0,t)=g_1(t),$$

Applying (2) on eq. (28), we get

$$T_{2g}^{\,c}\{\Psi_{tt}(\chi,t)\}-T_{2g}^{\,c}\{\Psi_{\chi\chi}(\chi,t)\}+aT_{2g}^{\,c}\{\Psi(\chi,t)\}=T_{2g}^{\,c}\{h(\chi,t)\}$$

$$\Rightarrow [iq_2(r)]^2 T_{2_g}^{\ c}\{\Psi(\chi,t)\} - iq_2(r)p_2(r)T_g^c\{\Psi(\chi,0)\} - p_2(r)T_g^c\{\Psi_t(\chi,0)\} - [iq_1(s)]^2 T_{2_g}^{\ c}\{\Psi(\chi,t)\} + iq_1(s)p_1(s)T_g^c\{\Psi(0,t)\} + p_1(s)T_g^c\{\Psi_\chi(0,t)\} + aT_{2_g}^{\ c}\{\Psi(\chi,t)\}T_{2_g}^{\ c}\{h(\chi,t)\},$$

$$T_{2_g}^{\ c}\{\Psi(\chi,t)\} = \frac{iq_2(r)p_2(r)T_g^c\{\Psi(\chi,0)\} + p_2(r)T_g^c\{\Psi_t(\chi,0)\} - iq_1(s)p_1(s)T_g^c\{\Psi(0,t)\} - p_1(s)T_g^c\{\Psi_\chi(0,t)\}}{[iq_2(r)]^2 - [iq_1(s)]^2 + a}. \quad (29)$$

With taking single SEJI integral transform to the initial conditions, we obtain

$$T_g^c\{\Psi(\chi,0)\} = T_g^c\{f_0(\chi)\}, \quad T_g^c\{\Psi_t(x,0)\} = T_g^c\{f_1(\chi)\}, \quad (30)$$
$$T_g^c\{\Psi(0,t)\} = T_g^c\{g_0(t)\}, \quad T_g^c\{\Psi_\chi(0,t)\} = T_g^c\{g_1(t)\}. \quad (31)$$

Substituting (37), (38) in (36), we get

$$T_{2_g}^{\ c}\{\Psi(\chi,t)\} = \frac{iq_2(r)p_2(r)T_g^c\{f_0(\chi)\} + p_2(r)T_g^c\{f_1(\chi)\} - iq_1(s)p_1(s)T_g^c\{g_0(t)\} - p_1(s)T_g^c\{g_1(t)\}}{[iq_2(r)]^2 - [iq_1(s)]^2 + a}. (32)$$

Example6:

To solve the linear homogeneous Klein-Gordon equation:

$$\Psi_{tt}(\chi,t) - \Psi_{\chi\chi}(\chi,t) - \Psi(\chi,t) = 0, \quad (33)$$

Subject to the following initial conditions

$$\Psi(\chi,0) = f_0(\chi) = 0, \quad \Psi_t(\chi,0) = f_1(\chi) = \sin(\chi),$$
$$\Psi(0,t) = g_0(t) = 0, \quad \Psi_\chi(0,t) = g_1(t) = t, \quad t \geq 0$$

we have

$$T_{2_g}^{\ c}\{h(\chi,t)\} = 0, \quad (34)$$

$$T_g^c\{f_0(\chi)\} = 0, \quad T_g^c\{f_1(\chi)\} = T_g^c\{\sin(\chi)\} = \frac{-p_1(s)}{[q_1(s)]^2 - 1}, \quad (35)$$

$$T_g^c\{g_0(t)\} = 0, \quad T_g^c\{g_1(t)\} = T_g^c\{t\} = \frac{(-i)^2 p_2(r)}{[q_2(r)]^2}, \quad (36)$$

Then, by substituting (34-36) in (12), we get

$$T_{2_g}^{\ c}\{\Psi(\chi,t)\} = \frac{p_2(r)\left[\frac{-p_1(s)}{[q_1(s)]^2 - 1}\right] - p_1(s)\left[\frac{(-i)^2 p_2(r)}{[q_2(r)]^2}\right]}{[iq_2(r)]^2 - [iq_1(s)]^2 - 1},$$
$$= \frac{p_1(s)p_2(r)}{[q_2(r)]^2([q_1(s)]^2 - 1)}.$$

The exact solution for the eq. (33) is

$$\Psi(\chi,t) = T_{2_g}^{\ c^{-1}}\left\{\frac{p_1(s)p_2(r)}{([q_1(s)]^2 - 1)[q_2(r)]^2}\right\} = T_{2_g}^{\ c^{-1}}\left\{\frac{-p_2(r)}{[q_2(r)]^2} \cdot \frac{-p_1(s)}{([q_1(s)]^2 - 1)}\right\}$$

By the formula 5.10, we get

$$T_{2_g}^{\ c^{-1}}\left\{\frac{-p_2(r)}{[q_2(r)]^2} \cdot \frac{-p_1(s)}{([q_1(s)]^2 - 1)}\right\} = T_g^{c^{-1}}\left\{\frac{(-i)^2 p_2(r)}{[q_2(r)]^2}\right\} T_g^{c^{-1}}\left\{\frac{-p_1(s)}{([q_1(s)]^2 - 1)}\right\},$$
$$\Psi(\chi,t) = t\sin(\chi).$$

8. Conclusion

In two-dimensional spaces, we developed a brand-new transform called the double SEJI integral transform that was derived from the SEJI integral transform. The convolution theorem and certain fundamental features connected to this new transform are demonstrated. We introduced some partial differential equations and used the recommended transform to find the solutions to these equations in order to demonstrate the efficiency of the transform. Future research on the behavior of the double SEJI integral transform and how higher order and fractional order partial differential equations might be affected by it are both essential.

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