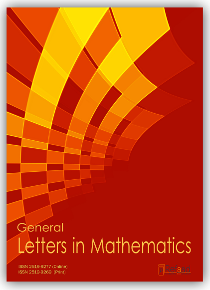

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Normal Subgroups and Quotient Groups in β-α-topological Group

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Abstract. In this paper, we studied the subgroup in β-α-topological group and several basic theorems are introduced. From normal subgroups, the study naturally follows to quotient groups, which are decomposition spaces topologically. The quotient group space of β-α-topological groups defined with the quotient topology. Also, the definition and some basic results of isomorphism are presented.

Keywords: *operations, α-open set, subgroups, β-α-topological group.*
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1 Introduction

Topological groups are objects that combine two separate structures-the structure of a topological space and the algebraic structure of a group-linked by the requirement that the group operations are continuous with respect to the underlying topology.

In 2013, operations defined on the family of α-open sets and α_γ-open were introduced by Ibrahim [11]. In [10], Khalaf and Ibrahim defined β-α-topological groups as a group G endowed with a topology. Also in [10], some results was given.

In this paper our aim is to develop the notion of β-α-topological group, we establish several theorems and properties related to normal subgroup of β-α-topological group.

2 Preliminaries

Let A be a subset of a topological space (G, τ). We denote the interior and the closure of a set A by $Int(A)$ and $Cl(A)$ respectively. A subset A of a topological space (G, τ) is called α-open [12] if $A \subseteq Int(Cl(Int(A)))$. The complement of an α-open set is called α-closed. The intersection of all α-closed sets containing A is called the α-closure of A and is denoted by $\alpha Cl(A)$. By $\alpha O(G, \tau)$, we denote the family of all α-open sets of G. An operation $\beta : \alpha O(G, \tau) \to P(G)$ [11] is a mapping satisfying the condition, $V \subseteq V^\beta$ for each $V \in \alpha O(G, \tau)$. We call the mapping β an operation on $\alpha O(G, \tau)$.

A subset A of G is called an α_β-open set [11] if for each point $x \in A$, there exists an α-open set U of G containing x such that $U^\beta \subseteq A$. The complement of an α_β-open set is said to be α_β-closed. We denote the set of all α_β-open sets of (G, τ) by $\alpha O(G, \tau)_\beta$. The α_β-closure [11] of a subset A of G with an operation β on $\alpha O(G)$ is denoted by $\alpha_\beta Cl(A)$ and is defined to be the intersection of all α_β-closed sets containing A. A point $x \in G$ is in αCl_β-closure [11] of a set $A \subseteq G$, if $U^\beta \cap A \neq \phi$ for each α-open set U containing x. The αCl_β-closure of A is denoted by $\alpha Cl_\beta(A)$.

The union of all α_β-open sets contained in A is called the α_β-interior of A and denoted by $\alpha_\beta Int(A)$ [3]. An operation β on $\alpha O(G,\tau)$ is said to be α-open [11] if for every α-open set U of $x \in G$, there, exists an α_β-open set V of G such that $x \in V$ and $V \subseteq U^\beta$. The operation $id : \alpha O(G,\tau) \to P(G)$ is defined by $id(V) = V$ for any set $V \in \alpha O(G,\tau)$ this operation is called the identity operation on $\alpha O(G,\tau)$ [11].

An operation $\beta : \alpha O(G) \to P(G)$ is said to be α-monotone on $\alpha O(G)$ [3] if for all $A, B \in \alpha O(G)$, $A \subseteq B$ implies $A^\beta \subseteq B^\beta$. An operation $\beta : \alpha O(G) \to P(G)$ is said to be α-idempotent on $\alpha O(G)$ [3] if $A^{\beta\beta} = A^\beta$ for all $A \in \alpha O(G)$.

Definition 2.1. *[4] Let S be any subset of G. An operation β from $\alpha O(G)$ to $P(G)$ is called α-stable with respect to S if β has the following two properties:*

1. *For any subset H of G, $U \cap H = V \cap H$ implies that $U^\beta \cap H = V^\beta \cap H$ for every $U, V \in \alpha O(G)$.*
2. *β induces an operation $\beta_S : P(S) \to P(S)$ such that $(U \cap S)^{\beta_S} = U^\beta \cap S$ for every $U \in \alpha O(G)$.*

Definition 2.2. *[2] A topological space (G,τ) is said to be α_β-regular if for each $x \in G$ and for each α-open set V in G containing x, there exists an α-open set U in G containing x such that $U^\beta \subseteq V$.*

Definition 2.3. *[1] A space G is said to be α-compact if every α-open cover of G has a finite subcover*

Definition 2.4. *[7] A subset A of a topological space (G,τ) is said to be α-β-compact of G if for every α-open cover $\{V_i : i \in I\}$ of A, there exists a finite subset I_0 of I such that $A \subseteq \bigcup\{V_i^\beta : i \in I_0\}$.*

Definition 2.5. *[9] Two subsets A and B of a topological space (G,τ) are called α_β-separated if $(\alpha_\beta Cl(A) \cap B) \cup (A \cap \alpha_\beta Cl(B)) = \phi$.*

Definition 2.6. *[9] A subset C of a space G is said to be α_β-disconnected if there are nonempty α_β-separated subsets A and B of G such that $C = A \cup B$, otherwise C is called α_β-connected.*

Definition 2.7. *[9] A set C is called maximal α_β-connected set if it is α_β-connected and if $C \subseteq D \subseteq G$ where D is α_β-connected, then $C = D$. A maximal α_β-connected subset C of a space G is called an α_β-component of G.*

Definition 2.8. *[5] A topological space (G,τ) with an operation β on $\alpha O(G)$ is said to be:*

1. *An α-β-T_1 space if for any two distinct points $x, y \in G$, there exist two α-open sets U and V containing x and y, respectively, such that $y \notin U^\beta$ and $x \notin V^\beta$.*
2. *An α-β-T_2 space if for any two distinct points $x, y \in G$, there exist two α-open sets U and V containing x and y, respectively, such that $U^\beta \cap V^\beta = \phi$.*

Definition 2.9. *[6] A space G is said to be weakly α_β-regular space, if for any α_β-closed set A and $x \notin A$, there exist α_β-open sets U, V such that $x \in U$, $A \subseteq V$ and $U \cap V = \phi$.*

Definition 2.10. *A function $f : (G_1,\tau) \to (G,\sigma)$ is said to be:*

1. *α-β-continuous [8] if for each point x in G and for each α-open set V of G containing $f(x)$, there exists an α-open set U of G_1 containing x such that $f(U) \subseteq V^\beta$.*
2. *α-(β_1,β)-continuous [8] if for each $x \in G_1$ and each α-open set V containing $f(x)$, there exists an α-open set U such that $x \in U$ and $f(U^{\beta_1}) \subseteq V^\beta$.*
3. *$\alpha_{(\beta_1,\beta)}$-continuous [11] if for each x of G_1 and each α_β-open set V containing $f(x)$, there exists an α_{β_1}-open set U such that $x \in U$ and $f(U) \subseteq V$.*
4. *$\alpha_{(\beta_1,\beta)}$-closed [11] if for any α_{β_1}-closed set A of (G_1,σ), $f(A)$ is α_β-closed in (G,τ).*
5. *$\alpha_{(\beta_1,\beta)}$-open [8] if for any α_{β_1}-open set A of (G_1,σ), $f(A)$ is α_β-open in (G,τ).*
6. *α-(β_1,β)-homeomorphic [8] if f is bijective, α-(β_1,β)-continuous and f^{-1} is α-(β,β_1)-continuous.*

Throughout this paper, $(G,*,\tau)$ and $(G_1,*,\sigma)$, or simply G and G_1, will denote groups $(G,*)$ and $(G_1,*)$ endowed with a topology τ and σ. The identity element of G is denoted by e. The operations $\beta : \alpha O(G) \to P(G)$ and $\beta_1 : \alpha O(G_1) \to P(G_1)$ are always operations defined on $\alpha O(G)$ and $\alpha O(G_1)$, respectively.

The operation $* : G \times G \to G$, $(x,y) \to x * y$ is called the multiplication mapping and sometimes denoted by m, and the inverse operation $G \to G$, $x \to x^{-1}$ is denoted by i.

Definition 2.11. *[10] Let $(G, *)$ be a group and τ be a topology on G. Then, the multiplication map is β-α-continuous in the first variable if and only if given $a, b \in G$ and $O \in \alpha O(G, \tau)$ such that $a * b \in O$, then there is $U \in \alpha O(G, \tau)$ with $a \in U$ and $U^\beta * b \subseteq O^\beta$. Similarly, multiplication is β-α-continuous in the second variable if and only if given $a, b \in G$ and $O \in \alpha O(G, \tau)$ such that $a * b \in O$, then there is $V \in \alpha O(G, \tau)$ with $b \in V$ and $a * V^\beta \subseteq O^\beta$.*

Definition 2.12. *[10] Let $(G, *)$ be a group and τ be a topology on G.*

1. *The inversion map is β-α-continuous if and only if given $a \in G$ and $O \in \alpha O(G, \tau)$ such that $a^{-1} \in O$, then there is $U \in \alpha O(G, \tau)$ with $a \in U$ and $U^{\beta^{-1}} \subseteq O^\beta$, where $U^{\beta^{-1}} = \{x^{-1} : x \in U^\beta\}$.*
2. *The multiplication is jointly β-α-continuous in both variables if and only if given $a, b \in G$ and $O \in \alpha O(G, \tau)$ such that $a * b \in O$, then there exist $U, V \in \alpha O(G, \tau)$ with $a \in U$, $b \in V$ and $U^\beta * V^\beta \subseteq O^\beta$.*
3. *A triple $(G, *, \tau)$ is called a β-α-topological group if and only if inversion is β-α-continuous and multiplication is jointly β-α-continuous in both variables.*

Theorem 2.13. *[10] Let $(G, *)$ be a group and τ be a topology on G. Then inversion is β-α-continuous and multiplication is jointly β-α-continuous in both variables if and only if for any elements a, b of G and α-open set O with $a * b^{-1} \in O$, there exist α-open sets U and V containing a and b respectively such that $U^\beta * V^{\beta^{-1}} \subseteq O^\beta$.*

Corollary 2.14. *[10] Suppose that the multiplication map is β-α-continuous in each variable. Let β be α-open, α-monotone and α-idempotent. If S is a semigroup, then $\alpha Cl_\beta(S)$ is also semigroup.*

Theorem 2.15. *[10] Let $(G, *)$ be a group, τ be a topology on G and (G, τ) be α_β-regular. If the multiplication map is β-α-continuous in the second variable. If S is a semigroup, then $\alpha_\beta Int(S)$ is also semigroup.*

Theorem 2.16. *[10] Let A and B be nonempty subsets of a β-α-topological group $(G, *, \tau)$ and β be identity. If A and B are α_β-connected, then $A * B$ is α_β-connected.*

Theorem 2.17. *[10] Let $(G, *)$ be a group and τ be a topology on G. If the multiplication map is β-α-continuous in each variable. If A is arbitrary and B is α_β-open, then $A * B$ and $B * A$ are α-open.*

Theorem 2.18. *[10] Let $(G, *, \tau)$ be a β-α-topological group. If A is α_β-closed and B is α-compact subsets of G, then $A * B$ and $B * A$ are α-closed.*

Theorem 2.19. *[10] Let $(G, *, \tau)$ be a β-α-topological group, (G, τ) be α_β-regular and β be α-monotone, α-left and α_β-left. Then, G is α-β-T_2 if and only if $\{e\}$ is α_β-closed.*

Theorem 2.20. *[10] Let $(G, *)$ be a group, τ be a topology on G and (G, τ) be α_β-regular. If the multiplication map is β-α-continuous in each variable. If S is a normal set algebraically, then $\alpha_\beta Int(S)$ and $\alpha_\beta Cl(S)$ are also normals.*

3 Quotient Group in β-α-topological Group

Recalling the following well known definition.

Definition 3.1. *A non empty subset S of the group G is a subgroup of G if $x * S = S = S * x$ for every $x \in S$. Equivalently, if for every $x, y \in S$, $x * y^{-1} \in S$.*
A subgroup S is a normal subgroup of G if $x * s * x^{-1} \in S$ for each $s \in S$ and each $x \in X$.

It is obvious that the group G and $\{e\}$ both are normal subgroups of G.

Theorem 3.2. *Let S be an α-open subgroup of a β-α-topological group $(G, *, \tau)$ and β be an α-monotone operation on $\alpha O(G)$ which is α-stable with respect to S. Then S is a β_S-α-topological group.*

Proof. We have to show that for each $x, y \in S$ and each α-open subset W in S with $x*y^{-1} \in W$, there exist α-open subsets U, V containing x and y respectively such that $U^{\beta_S} * (V^{\beta_S})^{-1} \subseteq W^{\beta_S}$. Since S is α-open in G, so there exists an α-open set L in G such that $W = L \cap S$ and since G is a β-α-topological group, then there are α-open sets A and B containing x and y respectively such that $A^\beta * (B^\beta)^{-1} \subseteq (L \cap S)^\beta$. The sets $U = A \cap S$ and $V = B \cap S$ are both in α-open in S. Also, $U^{\beta_S} * (V^{\beta_S})^{-1} = (A \cap S)^{\beta_S} * ((B \cap S)^{\beta_S})^{-1} = (A^\beta \cap S) * (B^\beta \cap S)^{-1} \subseteq A^\beta * (B^\beta)^{-1} \subseteq (L \cap S)^\beta \subseteq L^\beta$ implies that $U^{\beta_S} * (V^{\beta_S})^{-1} = (U^{\beta_S} * (V^{\beta_S})^{-1}) \cap S \subseteq L^\beta \cap S = (L \cap S)^{\beta_S} = W^{\beta_S}$. Hence, by Theorem 2.13, we obtain that S is a β_S-α-topological group.

Theorem 3.3. *Let $(G, *)$ be a group and τ be a topology on G. If the multiplication map is β-α-continuous in the second variable, then the following statements are true:*

1. *If H is an α_β-open subgroup of G, then it is α-closed in G.*
2. *If a subgroup H of G contains a non empty α_β-open set, then H is α-open in G.*

Proof.

1. Let H be an α_β-open subgroup of G. Then, by Theorem 2.17 every left coset $x * H$ is α-open. Thus, $Y = \bigcup_{x \in G \setminus H} x * H$ is also α-open as a union of α-open sets. Then $H = G \setminus Y$ and so H is α-closed.
2. Let B be a non empty α_β-open subset of G with $B \subseteq H$. Then, by Theorem 2.17, we have $h * B$ is α-open in G for any $h \in H$. Since H is a subgroup of G, so $h * B \subseteq H$ for all $h \in H$. Therefore, $H = \bigcup_{h \in H} h * B$ is also α-open as a union of α-open sets.

Theorem 3.4. *Let $(G, *, \tau)$ be a β-α-topological group and S a subgroup of G. If β is α-open, α-monotone and α-idempotent, then:*

1. *The set $\alpha Cl_\beta(S) = \alpha_\beta Cl(S)$ is a subgroup of G.*
2. *If (G, τ) is α_β-regular, then the set $S = \alpha_\beta Cl(S) = \alpha Cl_\beta(S)$ if and only if there exists an α_β-open set Q such that $Q \cap S = Q \cap \alpha_\beta Cl(S) \neq \phi$.*

Proof.

1. Since S is a semigroup, by Corollary 2.14, $\alpha Cl_\beta(S)$ is a semigroup. Since β is α-open and $e \in S$, so $e \in \alpha Cl_\beta(S) = \alpha_\beta Cl(S)$. Let $a \in \alpha_\beta Cl(S)$ and $O \in \alpha O(G, \tau)_\beta$ such that $a^{-1} \in O$, since f is α-(β, β)-continuous, then $O^{-1} \in \alpha O(G, \tau)_\beta$ and $a \in O^{-1}$. Thus there is $b \in S \cap O^{-1}$. Then $b^{-1} \in S \cap O$ and $a^{-1} \in \alpha_\beta Cl(S) = \alpha Cl_\beta(S)$. Therefore $\alpha Cl_\beta(S) = \alpha_\beta Cl(S)$ is a subgroup of G.
2. If $S = \alpha_\beta Cl(S)$, then G is an α_β-open set and $G \cap S = G \cap \alpha_\beta Cl(S) \neq \phi$.

   Conversely, let $x \in \alpha_\beta Cl(S)$ and $c \in S \cap Q$, then $x * Q * c^{-1} \in \alpha O(G, \tau) = \alpha O(G, \tau)_\beta$ such that $x = x * c * c^{-1} \in x * Q * c^{-1}$. Thus there is $s \in S \cap x * Q * c^{-1}$ which implies there is $q \in Q$ such that $s = x * q * c^{-1}$. Since $x * q * c^{-1}$ and c are elements of S, $x * q = x * q * c^{-1} * c \in S$. Hence, $q \in x^{-1} * S \subseteq \alpha_\beta Cl(S)$ and $q \in Q \cap \alpha_\beta Cl(S) = Q \cap S$. Thus $q \in S$. Since q^{-1} and $x * q$ are elements of S, so $x = (x * q) * q^{-1} \in S$. Therefore $\alpha_\beta Cl(S) \subseteq S$ and $S = \alpha_\beta Cl(S) = \alpha Cl_\beta(S)$.

Theorem 3.5. *Let $(G, *, \tau)$ be a β-α-topological group and S a subgroup of G. If (G, τ) is α_β-regular and $\alpha_\beta Int(S) \neq \phi$, then $\alpha_\beta Int(S) = S = \alpha_\beta Cl(S)$.*

Proof. Let $x \in \alpha_\beta Int(S)$, then there is an α_β-open set O containing x such that $O \subseteq S$. Thus $x^{-1} \in O^{-1} \subseteq S$ and $x^{-1} \in \alpha_\beta Int(S)$. Since $\alpha_\beta Int(S)$ is a semigroup by Theorem 2.15, $e = x * x^{-1} \in \alpha_\beta Int(S)$.

Let $x \in S$, then $x = x * e \in x * \alpha_\beta Int(S) \subseteq S$. Therefore $x \in \alpha_\beta Int(S)$ and $S = \alpha_\beta Int(S)$.

Let $x \in \alpha_\beta Cl(S)$, then $S = \alpha_\beta Int(S)$ and $e \in S$ imply $x * e \in x * S \in \alpha O(G, \tau) = \alpha O(G, \tau)_\beta$ such that $x \in x * S$. Since $S \cap x * S \neq \phi$, there is an $s_1 \in S \cap x * S$ such that $s_1 = x * s_2$ for some $s_2 \in S$. Then $x = s_1 * s_2^{-1} \in S$. Hence $\alpha_\beta Cl(S) \subseteq S$ and $S = \alpha_\beta Cl(S)$. Therefore, $\alpha_\beta Int(S) = S = \alpha_\beta Cl(S)$.

Theorem 3.6. *Let $(G, *, \tau)$ be a β-α-topological group and β be identity. If G_e is α_β-component subset of G such that $e \in G_e$, then G_e is α_β-closed normal subgroup.*

Proof. Since G_e is α_β-closed as it is an α_β-component. Let $a \in G_e$, then by Theorem 2.16, $a * G_e$ is α_β-connected. Thus there is an α_β-component C of G such that $a * G_e \subseteq C$. If $C \neq G_e$, then C and G_e are separated, but $a \in C \cap G_e$. Therefore $C = G_e$ and $a * G_e \subseteq G_e$. Let $b \in G_e$, since $a^{-1} * G_e$ is α_β-connected and $e \in a^{-1} * G_e$, so $a^{-1} * G_e \subseteq G_e$. Thus $a^{-1} * b \in G_e$ and $b \in a * G_e$. Hence $G_e \subseteq a * G_e$ and $G_e = a * G_e$. Similarly $G_e * a = G_e$. Therefore G_e is a subgroup.

Let $x \in G$, then $x * G_e * x^{-1}$ is α_β-connected and $e \in x * G_e * x^{-1}$ implies $x * G_e * x^{-1} \subseteq G_e$. Similarly $x^{-1} * G_e * x \subseteq G_e$, thus $G_e \subseteq x * G_e * x^{-1}$. Therefore $G_e = x * G_e * x^{-1}$ and G_e is normal.

Theorem 3.7. *Let $(G, *, \tau)$ be a β-α-topological group. If A is an α_β-closed subset of G, then the normalizer of A is α_β-closed subgroups of G.*

Proof. Let $N = \{x : x * A = A * x\}$ denote the normalizer of A and let $y \in N$, then $y * A = A * y$ implies $y^{-1} * A = A * y^{-1}$, thus $y^{-1} \in N$. If $x, y \in N$, then
$(x * y^{-1}) * A = x * (y^{-1} * A) = x * (A * y^{-1}) = (x * A) * y^{-1} = A * (x * y^{-1})$. Hence $x * y^{-1} \in N$ and N is a subgroup.

Let $r \in \alpha Cl_\beta(N)$ and let $r * a \in r * A$ for $a \in A$. Let $O \in \alpha O(G, \tau)$ such that $r * a * r^{-1} \in O$, then there are α-open sets U and V such that $r \in U$, $a \in V$ and $U^\beta * V^\beta * U^{\beta^{-1}} \subseteq O^\beta$. There is $n \in U^\beta \cap N$, thus $n * a * n^{-1} \in O^\beta$. Since $n * A = A * n$, so $n * a * n^{-1} \in A \cap O^\beta$. Thus $r * a * r^{-1} \in \alpha Cl_\beta(A) = A$, hence $r * a * r^{-1} \in A$. Then $(r * a * r^{-1}) * r = r * a \in A * r$ and $r * A \subseteq A * r$. Similarly $A * r \subseteq r * A$ and so $r * A = A * r$. Hence $r \in N$ and N is α_β-closed.

Theorem 3.8. *Let $(G, *, \tau)$ be a β-α-topological group and A be a subset of G. If G is α-β-T_2, then the centralizer of A is α_β-closed subgroups of G.*

Proof. Let $C = \{x : x * a = a * x$ for all $a \in A\}$ denote the centralizer of A. Let $y \in C$, then $y * a = a * y$ for every $a \in A$. Hence $a * y^{-1} = y^{-1} * a$ for every $a \in A$, thus $y^{-1} \in C$. Let $x, y \in C$ and $a \in A$, then
$(x * y^{-1}) * a = x * (y^{-1} * a) = x * (a * y^{-1}) = (x * a) * y^{-1} = a * (x * y^{-1})$. Thus $x * y^{-1} \in C$ and consequently C is a subgroup.

Let $p \in \alpha Cl_\beta(C)$. Let $a \in A$ and $O \in \alpha O(G, \tau)$ such that $p * a * p^{-1} \in O$, then there are α-open sets U and V with $p \in U$, $a \in V$ and $U^\beta * V^\beta * (U^\beta)^{-1} \subseteq O^\beta$. Since there is $x \in U^\beta \cap C$, $x * a * x^{-1} \in O^\beta$, but $x * a = a * x$, thus $a = x * a * x^{-1} \in O^\beta$. Therefore for every $O \in \alpha O(G, \tau)$ such that $p * a * p^{-1} \in O$, then $a \in O^\beta$. Suppose $p * a * p^{-1} \neq a$, since G is α-β-T_2, then there are α-open sets K and L such that $a \in K$, $p * a * p^{-1} \in L$ and $K^\beta \cap L^\beta = \phi$, but $p * a * p^{-1} \in L$ implies $a \in L^\beta$. This is a contradiction and thus, $a = p * a * p^{-1}$ and $p \in C$. Hence C is α_β-closed.

Theorem 3.9. *Let $(G, *, \tau)$ be a β-α-topological group and S a commutative subgroup of G. Suppose that β is α-open, α-monotone and α-idempotent. If G is α-β-T_2, then $\alpha Cl_\beta(S) = \alpha_\beta Cl(S)$ is a commutative subgroup of G.*

Proof. By Theorem 3.4 (1), $\alpha Cl_\beta(S) = \alpha_\beta Cl(S)$ is a subgroup.

Let $a \in S$ and $p \in \alpha Cl_\beta(S)$. Let $O \in \alpha O(G, \tau)$ such that $p * a * p^{-1} \in O$, then there are α-open sets U and V such that $p \in U$, $a \in V$ and $U^\beta * V^\beta * U^{\beta^{-1}} \subseteq O^\beta$. Since there is an $x \in U^\beta \cap S$, $x * a * x^{-1} \in O^\beta$, but $x * a \in S$ implies $x * a = a * x$, thus $a = x * a * x^{-1} \in O^\beta$. Therefore, if $p * a * p^{-1} \in O^\beta$ and O is an α-open set containing $p * a * p^{-1}$, then $a \in O^\beta$. Since G is α-β-T_2, this implies $p * a * p^{-1} = a$, thus $p * a * p^{-1} = a$ and $p * a = a * p$.

Let $p, x \in \alpha Cl_\beta(S)$ and suppose $p * x \neq x * p$. Then G being α-β-T_2 implies there are α-open sets O_1 and O_2 such that $p * x \in O_1$, $x * p \in O_2$ and $O_1^\beta \cap O_2^\beta = \phi$. Since $p * x \in O_1$, there exist α-open sets U_1 and V_1 such that $p \in U_1$, $x \in V_1$ and $U_1^\beta * V_1^\beta \subseteq O_1^\beta$. Similarly there are α-open sets U_2 and V_2 such that $p \in U_2$, $x \in V_2$ and $V_2^\beta * U_2^\beta \subseteq O_2^\beta$. Let $U = U_1 \cap U_2$ and $V = V_1 \cap V_2$, then U and V are α-open, $p \in U$, $x \in V$, $U^\beta * V^\beta \subseteq O_1^\beta$ and $V^\beta * U^\beta \subseteq O_2^\beta$. Since $p, x \in \alpha Cl_\beta(S)$, there are elements a and b of S such that $a \in U^\beta \cap S$ and $b \in V^\beta \cap S$. Thus $a * b \in U^\beta * V^\beta \subseteq O_1^\beta$ and $b * a \in V^\beta * U^\beta \subseteq O_2^\beta$, but $a * b = b * a$ implies $a * b \in O_1^\beta \cap O_2^\beta$. Since O_1^β and O_2^β were defined to be disjoint, the supposition is incorrect. Therefore $p * x = x * p$ and $\alpha Cl_\beta(S) = \alpha_\beta Cl(S)$ is commutative.

Theorem 3.10. *Let $(G, *, \tau)$ be a β-α-topological group and β be α-open, α-monotone and α-idempotent. If S a subgroup of G, K is a normal subgroup of S and $G = \alpha Cl_\beta(S)$, then $\alpha Cl_\beta(K) = \alpha_\beta Cl(K)$ is a normal subgroup of G.*

Proof. Since K is a subgroup of S, so K is a subgroup of G, thus by Theorem 3.4 (1), $\alpha Cl_\beta(K) = \alpha_\beta Cl(K)$ is a subgroup of G.

Let $x \in G$ and $x * y * x^{-1} \in x * \alpha Cl_\beta(K) * x^{-1}$ where $y \in \alpha Cl_\beta(K)$. Let $O \in \alpha O(G, \tau)$ such that $x * y * x^{-1} \in O$, then there are α-open sets U and V with $x \in U$, $y \in V$ and $U^\beta * V^\beta * U^{\beta^{-1}} \subseteq O^\beta$. Since $G = \alpha Cl_\beta(S)$, there is $s \in S \cap U^\beta$ and $y \in \alpha Cl_\beta(K)$ implies there is $k \in V^\beta \cap K$. Thus $s * k * s^{-1} \in O^\beta$. Since K is normal with respect to S, $s * k * s^{-1} \in O^\beta \cap K$, hence $x * y * x^{-1} \in \alpha Cl_\beta(K)$ and $x * \alpha Cl_\beta(K) * x^{-1} \subseteq \alpha Cl_\beta(K)$. Similarly $x^{-1} * \alpha Cl_\beta(K) * x \subseteq \alpha Cl_\beta(K)$ and $\alpha Cl_\beta(K) \subseteq x * \alpha Cl_\beta(K) * x^{-1}$. Therefore $\alpha Cl_\beta(K) = x * \alpha Cl_\beta(K) * x^{-1}$ for all $x \in G$. Hence, K is normal in G.

Definition 3.11. *Let S be a normal subgroup of a group G. Consider the family $G/S = \{g * S : g \in G\}$, consisting of all left cosets $g * S$ of S in G. We define the binary operation $*$ on G/S by the formula $g * S * p * S = g * p * S$ for all $g, p \in G$. The operation $*$ makes G/S a group whose neutral element is S and where the inverse of an element*

$g * S$ is $g^{-1} * S$.

Consider the mapping $\pi : G \to G/S$ defined by $\pi(g) = g * S$, for each $g \in G$, then this mapping is a group homomorphism and for each $g \in G$, we have $\pi^{-1}(\pi(g)) = g * S$.

Let $(G, *, \tau)$ be a β-α-topological group. Denote by τ' the topology of G/S and it is called the quotient topology of the quotient group G/S of the group G. In the set G/S, we define a family τ' and $\alpha O(G/S, \tau')$ of subsets as follows:
$\tau' = \{O \subseteq G/S : \pi^{-1}(O) \in \tau\}$ *and*
$\alpha O(G/S, \tau') = \{O \subseteq G/S : \pi^{-1}(O) \in \alpha O(G, \tau)\}$.

From the operation β which is defined on $\alpha O(G, \tau)$, we define the operation $\beta_{G/S} : \alpha O(G/S, \tau') \to P(G/S)$ as follows:
$(\pi(U))^{\beta_{G/S}} = \pi(U^\beta)$ *for every $U \in \alpha O(G, \tau)$ and $\pi(U) \in \alpha O(G/S, \tau')$.*

Example 3.12. *Consider the β-α-topological group $(Z_{12}, +_{12}, \tau)$, where $\tau = \{\phi, Z_{12}, \{0, 1, 2, 3, 4, 5\}, \{6, 7, 8, 9, 10, 11\}\}$ and for each $A \in \alpha O(Z_{12}, \tau)$, we define β on $\alpha O(Z_{12}, \tau)$ by $A^\beta = Z_{12}$. Let $S = \{0, 3, 6, 9\}$, so $Z_{12}/S = \{S, 1 +_{12} S, 2 +_{12} S\}$. Then, $\alpha O(Z_{12}/S, \tau') = \{\phi, Z_{12}/S\}$.*

Theorem 3.13. *Let $(G, *, \tau)$ be a β-α-topological group and let S be a normal subgroup of G. If (G, τ) is α_β-regular, then $(G/S, *, \tau')$ is a $\beta_{G/S}$-α-topological group.*

Proof. First we show that $\pi(U) \in \alpha O(G/S, \tau')$ for every $U \in \alpha O(G, \tau)$. By the definition of the topology $\alpha O(G/S, \tau')$, we have that $\pi(U) \in \alpha O(G/S, \tau')$ when $\pi^{-1}(\pi(U)) \in \alpha O(G, \tau)$. For every $g \in G$, we have $\pi^{-1}(\pi(g)) = g * S$ from this it follows that $\pi^{-1}(\pi(U)) = \bigcup_{g \in U} g * S = U * S$. By Theorem 2.17, we have $U * S \in \alpha O(G, \tau)$ whenever $U \in \alpha O(G, \tau) = \alpha O(G, \tau)_\beta$ because (G, τ) is α_β-regular. Hence, we have $\pi^{-1}(\pi(U)) \in \alpha O(G, \tau)$ and so $\pi(U) \in \alpha O(G/S, \tau')$, for every $U \in \alpha O(G, \tau)$.

Next we show that the multiplication mapping $(a, b) \to a * b$ is jointly $\beta_{G/S}$-α-continuous in both variables $(G/S, \tau') \times (G/S, \tau') \to (G/S, \tau')$.

Let $O \in \alpha O(G/S, \tau')$ and let $a, b \in G/S$ such that $a * b \in O$. Let $x, y \in G$ satisfy $a = \pi(x)$ and $b = \pi(y)$. Since π is homomorphism, so $\pi(x * y) = \pi(x) * \pi(y) = a * b \in O$ and thus $x * y \in \pi^{-1}(O)$. Since $O \in \alpha O(G/S, \tau')$, we have $\pi^{-1}(O) \in \alpha O(G, \tau)$. Since $(G, *, \tau)$ is a β-α-topological group and $x * y \in \pi^{-1}(O) \in \alpha O(G, \tau)$, there exist $U, V \in \alpha O(G, \tau)$ such that $x \in U$, $y \in V$ and $U^\beta * V^\beta \subseteq (\pi^{-1}(O))^\beta$. Again since π is a homomorphism, we have $\pi(U^\beta * V^\beta) = \pi(U^\beta) * \pi(V^\beta)$. Since $U^\beta * V^\beta \subseteq (\pi^{-1}(O))^\beta$, we have $\pi(U^\beta * V^\beta) \subseteq \pi((\pi^{-1}(O))^\beta)$ and therefore $\pi(U^\beta) * \pi(V^\beta) \subseteq \pi((\pi^{-1}(O))^\beta)$ implies $(\pi(U))^{\beta_{G/S}} * (\pi(V))^{\beta_{G/S}} \subseteq (\pi(\pi^{-1}(O)))^{\beta_{G/S}} = O^{\beta_{G/S}}$. Hence, we have that $\pi(U) \in \alpha O(G/S, \tau')$ and $\pi(V) \in \alpha O(G/S, \tau')$. Since $a = \pi(x) \in \pi(U)$ and $b = \pi(y) \in \pi(V)$, we have shown that the multiplication mapping is jointly $\beta_{G/S}$-α-continuous in both variables.

Now, we have to show that the inversion mapping $a \to a^{-1}$ is $\beta_{G/S}$-α-continuous $(G/S, \tau') \to (G/S, \tau')$.

Let $a \in G/S$ and let $O \in \alpha O(G/S, \tau')$ such that $a^{-1} \in O$. let $x \in G$ such that $a^{-1} = \pi(x^{-1})$ and $a = \pi(x)$. Then $\pi(x^{-1}) = a^{-1} \in O$ and thus $x^{-1} \in \pi^{-1}(O)$. Since $\pi^{-1}(O) \in \alpha O(G, \tau)$, there is an α-open set U such that $x \in U$ and $(U^\beta)^{-1} \subseteq (\pi^{-1}(O))^\beta$. Now $\pi(x) = a \in \pi(U)$, and $\pi(U) \in \alpha O(G/S, \tau')$. Since π is a homomorphism, so $\pi(U^{\beta^{-1}}) \subseteq \pi(\pi^{-1}(O))^\beta$ implies $\pi(U^\beta)^{-1} \subseteq \pi(\pi^{-1}(O))^\beta$ and hence $(\pi(U))^{\beta_{G/S}})^{-1} \subseteq (\pi(\pi^{-1}(O)))^{\beta_{G/S}} = O^{\beta_{G/S}}$. Therefore the inversion is $\beta_{G/S}$-α-continuous and hence $(G/S, *, \tau')$ is a $\beta_{G/S}$-α-topological group.

Example 3.14. *Let $(Z_6, +_6)$ be a group, τ be the discrete topology on Z_6 and $S = \{0, 3\}$, then*
$\alpha O(Z_6/S, \tau') = \{\phi, Z_6/S, \{S\}, \{1 +_6 S\}, \{2 +_6 S\}, \{S, 1 +_6 S\}, \{S, 2 +_6 S\}, \{1 +_6 S, 2 +_6 S\}\}$.
Now, for each $A \in \alpha O(Z_6, \tau)$, we define β on $\alpha O(Z_6, \tau)$ by

$$A^\beta = \begin{cases} A & \text{if } A \text{ is singleton set,} \\ Z_6 & \text{otherwise.} \end{cases}$$

Then, $\{S\}^{\beta_{Z_6/S}} = \pi(\{0, 3\})^{\beta_{Z_6/S}} = \pi(\{0, 3\}^\beta) = \pi(Z_6) = Z_6/S$,
$\{1 +_6 S\}^{\beta_{Z_6/S}} = \pi(\{1, 4\})^{\beta_{Z_6/S}} = \pi(\{1, 4\}^\beta) = \pi(Z_6) = Z_6/S$,
$\{2 +_6 S\}^{\beta_{Z_6/S}} = \pi(\{2, 5\})^{\beta_{Z_6/S}} = \pi(\{2, 5\}^\beta) = \pi(Z_6) = Z_6/S$,
$\{S, 1 +_6 S\}^{\beta_{Z_6/S}} = \pi(\{0, 1, 3, 4\})^{\beta_{Z_6/S}} = \pi(\{0, 1, 3, 4\}^\beta) = \pi(Z_6) = Z_6/S$,
$\{S, 2 +_6 S\}^{\beta_{Z_6/S}} = \pi(\{0, 2, 3, 5\})^{\beta_{Z_6/S}} = \pi(\{0, 2, 3, 5\}^\beta) = \pi(Z_6) = Z_6/S$ *and*
$\{1 +_6 S, 2 +_6 S\}^{\beta_{Z_6/S}} = \pi(\{1, 2, 4, 5\})^{\beta_{Z_6/S}} = \pi(\{1, 2, 4, 5\}^\beta) = \pi(Z_6) = Z_6/S$.
Therefore, $(Z_6, +_6, \tau)$ is β-α-topological group and $(Z_6/S, +_6, \tau')$ is a $\beta_{Z_6/S}$-α-topological group.

Theorem 3.15. *If (G,τ) is α_β-regular and $(G/S,\tau^{'})$ is $\alpha_{\beta_{G/S}}$-regular, then the natural homomorphism π from the β-α-topological group G to its quotient group G/S is an α-$\beta_{G/S}$-continuous, α-$(\beta,\beta_{G/S})$-continuous and $\alpha_{(\beta,\beta_{G/S})}$-continuous mapping.*

Proof. Since (G,τ) is α_β-regular and $(G/S,\tau^{'})$ is $\alpha_{\beta_{G/S}}$-regular, then $\alpha O(G,\tau)=\alpha O(G,\tau)_\beta$ and $\alpha O(G/S,\tau^{'})=\alpha O(G/S,\tau^{'})_{\beta_{G/S}}$.

Let $O\in\alpha O(G/S,\tau^{'})_{\beta_{G/S}}$, then $\pi^{-1}(O)$ is α_β-open in G by Definition 3.11, therefore π is α-$\beta_{G/S}$-continuous, α-$(\beta,\beta_{G/S})$-continuous and $\alpha_{(\beta,\beta_{G/S})}$-continuous.

Theorem 3.16. *If $(G/S,\tau^{'})$ is $\alpha_{\beta_{G/S}}$-regular, then the natural homomorphism π from a β-α-topological group G to its quotient group G/S is an $\alpha_{(\beta,\beta_{G/S})}$-open mapping.*

Proof. Let $U\in\alpha O(G,\tau)_\beta$, then $U\in\alpha O(G,\tau)$. Since in the beginning of the proof of Theorem 3.13, we showed that $\pi(U)\in\alpha O(G/S,\tau^{'})$ whenever $U\in\alpha O(G,\tau)$. As a consequence, π is an $\alpha_{(\beta,\beta_{G/S})}$-open mapping.

Theorem 3.17. *Let S be an α-compact normal subgroup of a β-α-topological group G. If (G,τ) is α_β-regular and $(G/S,\tau^{'})$ is $\alpha_{\beta_{G/S}}$-regular, then the natural homomorphism $\pi: G\to G/S$ is an $\alpha_{(\beta,\beta_{G/S})}$-closed mapping.*

Proof. Let F be an α_β-closed subset of G. By Theorem 2.18, the set $F*S$ is α-closed in G and hence the set $G\setminus(F*S)$ is α-open. Since π is $\alpha_{(\beta,\beta_{G/S})}$-open mapping, the subset $\pi(G\setminus(F*S))$ of G/S is $\alpha_{\beta_{G/S}}$-open. We have that $F*S=\pi^{-1}(\pi(F))$ and hence that $G\setminus(F*S)=\pi^{-1}(G/S\setminus\pi(F))$. It follows that $\pi(G\setminus(F*S))=G/S\setminus\pi(F)$. By the foregoing, the set $G/S\setminus\pi(F)$ is $\alpha_{\beta_{G/S}}$-open in G/S, hence $\pi(F)$ is $\alpha_{\beta_{G/S}}$-closed in G/S.

Theorem 3.18. *Let S be a normal subgroup of a β-α-topological group G and β be α-monotone, α-left and α_β-left. Suppose (G,τ) is α_β-regular and $(G/S,\tau^{'})$ is $\alpha_{\beta_{G/S}}$-regular. Then, G/S is α-$\beta_{G/S}$-T_1 if and only if S is α_β-closed.*

Proof. Let G/S be α-$\beta_{G/S}$-T_1, then the subset $\{S\}$ of G/S is $\alpha_{\beta_{G/S}}$-closed, from this it follows by α-$(\beta,\beta_{G/S})$-continuity of the mapping π, that the subset $S=\pi^{-1}(\{S\})$ of G is α_β-closed.

Conversely, assume S is α_β-closed in G, then $G\setminus S$ is α_β-open and it follows from $\alpha_{(\beta,\beta_{G/S})}$-openness of the mapping π that the subset $\pi(G\setminus S)$ of G/S is $\alpha_{\beta_{G/S}}$-open, since $G/S\setminus\{S\}=\pi(G\setminus S)$, then the subset $\{S\}$ of G/S is $\alpha_{\beta_{G/S}}$-closed and by Theorem 2.19, G/S is α-$\beta_{G/S}$-T_2. Therefore G/S is α-$\beta_{G/S}$-T_1.

Theorem 3.19. *Let S be a normal subgroup of a β-α-topological group G. Suppose (G,τ) is α_β-regular and $(G/S,\tau^{'})$ is $\alpha_{\beta_{G/S}}$-regular.*

1. *If G is α_β-connected, then G/S is $\alpha_{\beta_{G/S}}$-connected.*
2. *If G is α-compact, then G/S is α-$\beta_{G/S}$-compact.*

Proof.

1. Assume G is α_β-connected, then π being $\alpha_{(\beta,\beta_{\frac{G}{S}})}$-continuous implies $\pi(G)=G/S$ is $\alpha_{\beta_{G/S}}$-connected.
2. Since π is α-$\beta_{G/S}$-continuous and onto and G is α-compact, then G/S is α-$\beta_{G/S}$-compact.

Theorem 3.20. *Let $(G,*,\tau)$ be a β-α-topological group, (G,τ) be α_β-regular and $(G/S,\tau^{'})$ be $\alpha_{\beta_{\frac{G}{S}}}$-regular. If β is α-open, α-monotone, α-idempotent α-left and α_β-left, then $(G/\alpha Cl_\beta(\{e\}),*,\tau^{'})$ is an α-β^G_S-T_2 β^G_S-α-topological group.*

Proof. Since $\{e\}$ is a normal subgroup, by Theorems 2.20 and 3.4 (1) , $\alpha Cl_\beta(\{e\})$ is a normal subgroup, thus by Theorem 3.13, $G/\alpha Cl_\beta(\{e\})$ is a $\beta_{G/S}$-α-topological group. Since $\alpha Cl_\beta(\{e\})$ is α_β-closed, by Theorem 3.18, $G/\alpha Cl_\beta(\{e\})$ is α-$\beta_{G/S}$-T_1. Therefore by Theorem 2.19, $G/\alpha Cl_\beta(\{e\})$ is α-$\beta_{G/S}$-T_2.

Proposition 3.1. *Let $(G,*,\tau)$ be a β-α-topological group, (G,τ) be α_β-regular, $(G/S,\tau^{'})$ be $\alpha_{\beta_{G/S}}$-regular, S be a normal subgroup of G, π be the natural mapping of G onto G/S and let $U and V$ be an α-open subsets of G such that $e\in U$, $e\in V$ and $V^{-1}*V\subseteq U$. Then $\alpha_{\beta_{G/S}}Cl(\pi(V))\subseteq\pi(U)$.*

Proof. Take any $x \in G$ such that $\pi(x) \in \alpha_{\beta_{G/S}}Cl(\pi(V))$. Since $V * x$ is an α-open set containing x and the mapping π is $\alpha_{(\beta,\beta_{G/S})}$-open, then $\pi(V * x)$ is an $\alpha_{\beta_{G/S}}$-open set containing $\pi(x)$. Therefore, $\pi(V * x) \cap \pi(V) \neq \phi$. It follows that, for some $a \in V$ and $b \in V$, we have $\pi(a * x) = \pi(b)$, that is, $a * x = b * h$, for some $h \in S$. Hence, $x = (a^{-1} * b) * h \in U * S$, since $a^{-1} * b \in V^{-1} * V \subseteq U$. Therefore, $\pi(x) \in \pi(U * S) = \pi(U)$.

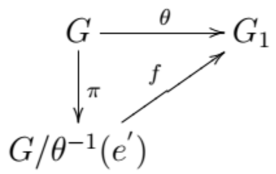
Corollary 3.21. *Let $(G, *, \tau)$ be a β-α-topological group, $(G/S, \tau^{'})$ be $\alpha_{\beta_{\frac{G}{S}}}$-regular, S be a normal subgroup of G and π be the natural mapping of G onto G/S. If $f : G/S \to G_1$ is a mapping such that the composition $f \circ \pi$ is α-(β, γ)-continuous and γ is an α-open operation on $\alpha O(G_1)$, then f is α-$(\beta_{G/S}, \gamma)$-continuous.*

Proof. Let W be an α_γ-open set in G_1. Since $f \circ \pi$ is α-(β, γ)-continuous, then $(f \circ \pi)^{-1}(W) = \pi^{-1}(f^{-1}(W))$ is α_β-open in G. Since by Theorem 3.16, π is an $\alpha_{(\beta,\beta_{\frac{G}{S}})}$-open mapping, then $\pi(\pi^{-1}(f^{-1}(W))) = f^{-1}(W)$ is $\alpha_{\beta_{G/S}}$-open in G/S. Therefore, f is α-$(\beta_{G/S}, \gamma)$-continuous.

Definition 3.22. *Let G be β-α-topological group and G_1 be γ-α-topological group. If there exists a homomorphism function $f : G \to G_1$ such that f is α-(β, γ)-homeomorphism, then G is said to be α-(β, γ)-isomorphic to G_1, and the function f is said to be α-(β, γ)-isomorphism.*

Theorem 3.23. *(The fundamental homomorphism theorem)*
Let $(G, *, \tau)$ be a β-α-topological group and (G, τ) be α_β-regular. If θ is an α-(β, γ)-continuous homomorphism of G onto G_1, γ is an α-open operation on $\alpha O(G_1)$ and $(G/\theta^{-1}(e^{'}), \tau^{'})$ is $\alpha_{\beta_{G/\theta^{-1}(e^{'})}}$-regular, then $G/\theta^{-1}(e^{'})$ is α-$(\beta_{G/\theta^{-1}(e^{'})}, \gamma)$-iseomorphic to G_1 if and only if θ is $\alpha_{(\beta,\gamma)}$-open, where $e^{'}$ is identity element of G_1.

Proof. Since θ is a homomorphism, so obviously $\theta^{-1}(e^{'})$ is a normal subgroup of G. Let π be the natural map from G to $G/\theta^{-1}(e^{'})$. Assume θ is $\alpha_{(\beta,\gamma)}$-open. Now, we have to show that $G/\theta^{-1}(e^{'})$ and G_1 are α-$(\beta_{G/\theta^{-1}(e^{'})}, \gamma)$-iseomorphic. Define $f : G/\theta^{-1}(e^{'}) \to G_1$ as $f \circ \pi = \theta$, thus



Let $y \in G_1$. Since θ is onto, there is an $x \in G$ such that $\theta(x) = y$. Let $\overline{x} = \pi(x)$ for $\overline{x} \in G/\theta^{-1}(e^{'})$, then $y = \theta\pi^{-1}(\overline{x}) = f(\overline{x})$. Thus f is onto.

To show that f is one to one, suppose that $f(a * \theta^{-1}(e^{'})) = f(b * \theta^{-1}(e^{'}))$, where $\pi(a) = a * \theta^{-1}(e^{'})$ and $\pi(b) = b * \theta^{-1}(e^{'})$ for $a, b \in G$. Then $f(\pi(a)) = f(\pi(b))$ implies $\theta(a) = \theta(b)$, so

$e^{'} = \theta^{-1}(a) * \theta(b) = \theta(a^{-1}) * \theta(b) = \theta(a^{-1} * b)$, used θ is homomorphism.

Thus, $a^{-1} * b \in \theta^{-1}(e^{'})$ implies that $a * \theta^{-1}(e^{'}) = b * \theta^{-1}(e^{'})$. Thus f is one to one.

Now, $f[(a*\theta^{-1}(e^{'}))*(b*\theta^{-1}(e^{'}))] = f[(a*b)*\theta^{-1}(e^{'})] = f(\pi(a*b)) = \theta(a*b) = \theta(a)*\theta(b) = f(\pi(a))*f(\pi(b)) = f(a*\theta^{-1}(e^{'}))*f(b*\theta^{-1}(e^{'}))$. Thus f is homomorphism.

Let O be an α_γ-open set in G_1. Since θ is α-(β, γ)-continuous and π is $\alpha_{(\beta,\beta_{G/\theta^{-1}(e^{'})})}$-open, then $\pi\theta^{-1}(O) = f^{-1}(O)$ is $\alpha_{\beta_{G/\theta^{-1}(e^{'})}}$-open in $G/\theta^{-1}(e^{'})$. Thus f is α-$(\beta_{G/\theta^{-1}(e^{'})}, \gamma)$-continuous.

Let O be an $\alpha_{\beta_{G/\theta^{-1}(e^{'})}}$-open set in $G/\theta^{-1}(e^{'})$. Since π is α-$(\beta, \beta^{G}_{\theta^{-1}(e^{'})})$-continuous and O is $\alpha_{\beta_{G/\theta^{-1}(e^{'})}}$-open, $\theta\pi^{-1}(O) = f(O)$ is α_γ-open in G_1. Thus f^{-1} is α-$(\gamma, \beta_{G/\theta^{-1}(e^{'})})$-continuous. Therefore f is an α-$(\beta_{G/\theta^{-1}(e^{'})}, \gamma)$-iseomorphism.

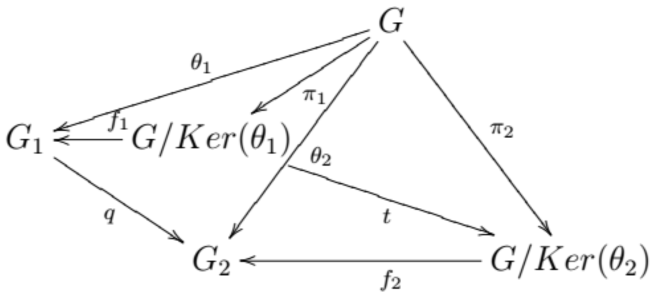
Conversely, assume f is an α-$(\beta_{G/\theta^{-1}(e^{'})}, \gamma)$-iseomorphism such that $\theta = f\pi$. Let O be an α_β-open set in G, since π is α-$(\beta, \beta_{G/\theta^{-1}(e^{'})})$-open and f^{-1} is α-$(\gamma, \beta_{G/\theta^{-1}(e^{'})})$-continuous, then $f\pi(O) = \theta(O)$ is α_γ-open in G_1. Hence θ is $\alpha_{(\beta,\gamma)}$-open.

Corollary 3.24. *Let $f : (G, *, \tau) \to (G_1, *, \tau_1)$ be an α-(β, γ)-isomorphism. If (G, τ) is α_β-regular, (G_1, τ_1) is α_γ-regular and $G_1/f(N)$ is $\alpha_{\beta_{G_1/f(N)}}$-regular for every normal subgroup N of G, then the quotient group G/N is α-$(\beta_{G/N}, \gamma_{G_1/f(N)})$-isomorphic to $G_1/f(N)$.*

Proof. Obviously $f(N)$ is a normal subgroup of G_1 and the surjective quotient homomorphism $q : G_1 \to G_1/f(N)$ is α-$(\gamma, \gamma_{G_1/f(N)})$-continuous and $\alpha_{(\gamma,\gamma_{G_1/f(N)})}$-open by Theorems 3.15 and 3.16. Therefore, the composition $h = q\circ f : G \to G_1/f(N)$ is a surjective α-$(\beta, \gamma_{G_1/f(N)})$-continuous and α-$(\beta, \gamma_{G_1/f(N)})$-open homomorphism with $Ker(h) = N$. Therefore, by Theorem 3.23, G/N is α-$(\beta_{G/N}, \gamma_{G_1/f(N)})$-iseomorphic to $G_1/f(N)$.

Proposition 3.2. *Let G, G_1 and G_2 be β-α-topological, β_1-α-topological and β_2-α-topological abelian groups respectively and (G, τ) be α_β-regular. Let $\theta_i : G \to G_i$, $i = 1, 2$, be α-(β, β_i)-continuous surjective $\alpha_{(\beta,\beta_i)}$-open homomorphisms and β_i be α-open. If $G/Ker(\theta_i)$ is $\alpha_{\beta_{G/Ker(\theta_i)}}$-regular and $Ker(\theta_1) \subseteq Ker(\theta_2)$, then there exists an α-(β_1, β_2)-continuous homomorphism $q : G_1 \to G_2$ such that $\theta_2 = q \circ \theta_1$.*

Proof. Assume that $Ker(\theta_1) \subseteq Ker(\theta_2)$ holds. By Theorem 3.23, there exists an α-$(\beta_{G/Ker(\theta_i)}, \beta_i)$-iseomorphisms $f_i : G/Ker(\theta_i) \to G_i$ such that $\theta_i = f_i \circ \pi_i$, where $\pi_i : G \to G/Ker(\theta_i)$ is the natural α-$(\beta, \beta_{G/Ker(\theta_i)})$-continuous $\alpha_{(\beta,\beta_{G/Ker(\theta_i)})}$-open homomorphisms for $i = 1, 2$. As $Ker(\theta_1) \subseteq Ker(\theta_2)$, we get an α-$(\beta_{G/Ker(\theta_1)}, \beta_{G/Ker(\theta_2)})$-continuous homomorphism t as in the following diagram:



Obviously $q = f_2 \circ t \circ f_1^{-1}$ works.

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