

An Accurate Approximate Solutions of Multipoint Boundary Value Problems

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Abstract. The main objective of this paper is to obtain a new accurate approximate solutions for a kind of ordinary differential equations called multipoint boundary value problems by using simple modification of optimal homotopy asymptotic method (OHAM). This procedure is a well-performance for calculating a better approximate solutions using one-order of approximation comparing with other methods which need higher order of approximations to gives the same results. Some examples are presented to testify the accuracy and applicability of this procedure. Comparisons are made between the present procedure and the other methods.

Keywords: *Multi-point boundary value problems , homotopy asymptotic method, Error estimate, ordinary differential equations.*

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1 Introduction

Multi-point boundary value problems (BVPs) appear in a many fields of applied mathematics and physics. For instance, the vibrations of a guy wire of uniform cross section composed of N parts different densities can be described by a multi-point BVP which was explained by Moshiinsky [1]. In fact, Multi-point BVPs arise in the mathematical modelling of viscoelastic and inelastic flows, deformation of beams and plate deflection theory [2]. Many approximated analytical or numerical methods have been used to find solutions of multi-point BVPs, Urabe [3] applied Chebyshev series to approximate solutions of nonlinear first-order multi-point BVPs, An efficient technique to find semi-analytical solutions for higher order multi-point boundary value problems is presented to solve general multi-point BVPs by Kheybari and Darvishi [4], Also, based on the differential transform method an efficient algorithm was successfully applied to obtain approximate solution of multi-point boundary value problems by Xie [5].

Real world physical problems are generally described by differential equations especially BVPs or Multipoint BVPs, various numerical or approximated method were utilized for solving these type of differential equations, like Finite Difference Method [6] and homotopy perturbation and variational iteration method [7], A six-step Block Unification [8] and so on. Many researchers have shown a great deal of interest on the approximate analytical solution for a wide classes of differential equations in the last few years using different procedures, one of the well-known powerful and efficient procedures for solving different types of differential equations is OHAM. In 2008, Marinca and Herisanu suggested the so-called optimal homotopy asymptotic method (OHAM) based on the homotopy equation

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in a series of papers [9, 10, 11] for the approximate solutions of nonlinear problems. This procedure give us with a convenient way to control the convergence of approximation series and demonstrates its validity and potential efficiency to solve a wide class of problems in applied science and engineering and also valid for small parameters. In the last few years, OHAM and its modifications has been applied successfully to solve many types of differential equations [12, 13, 14, 15, 16, 17, 18, 19, 20, 21].

In this paper, we will expand the application of OHAM by using simple modification to obtain a new accurate approximate analytic solution of multipoint boundary value problems throughout only one-order of approximation and comparing it with higher order of approximation using the same method [21]. The procedure is directly applied without any linearization and discretizations or splitting the non-homogeneous term. The structure of this paper is formulated as follows: Section 2 is devoted to the analysis of the proposed method, in Section 3, two examples are employed to illustrate the accuracy and computational efficiency of this procedure, and finally, conclusions are given in the last section.

2 Method of Solution

To explain the basic idea of OHAM [9, 22], we consider the following differential equation

$$L(u(x)) + g(x) + N(u(x)) = 0, \quad B\left(u, \frac{du}{dx}\right) = 0, \quad (1)$$

where L is the chosen linear operator, N is the linear or nonlinear operator, $u(x)$ is an unknown function, x denotes an independent variable, $g(x)$ is a known function and B is a boundary operator.

According to the basic idea OHAM we construct a homotopy $h(v(x, p), p) : R \times [0, 1] \rightarrow R$ which satisfies

$$\begin{aligned} (1-p)[L(v(x, p)) - u_0(x)] &= H(p)[L(v(x, p)) + g(x) + N(v(x, p))], \\ B\left(v(x, p), \frac{\partial v(x, p)}{\partial x}\right) &= 0, \end{aligned} \quad (2)$$

where $x \in R$ and $p \in [0, 1]$ is an embedding parameter, $H(p)$ is a nonzero auxiliary function for $p \neq 0$, $H(0) = 0$ and $v(x, p)$ is an unknown function. Obviously, when $p = 0$ and $p = 1$ it holds that $v(x, 0) = u_0(x)$ and $v(x, 1) = u(x)$ respectively. Thus, as p varies from 0 to 1, the solution $v(x, p)$ approaches from $u_0(x)$ to $u(x)$ where $u_0(x)$ is the initial guess that satisfies the linear operator and the boundary conditions

$$L(u_0(x)) = 0, \quad B\left(u_0, \frac{du_0}{dx}\right) = 0. \quad (3)$$

The auxiliary function $H(p)$ will be chosen in the following form

$$H(p) = \sum_{m=1}^k (C_m * x^{m-1}) * p, \quad (4)$$

where C_1, C_2, C_3, \dots are called the convergent control parameters which can be determined later.

To get an approximate solution, we expand $v(x, p, C_i)$ in Taylor's series about p in the following manner,

$$v(x, p, C_i) = u_0(x) + \sum_{k=1}^{\infty} u_k(x, C_1, C_2, \dots, C_k) p^k. \quad (5)$$

Substituting Eq.(5) into Eq.(2) and equating the coefficient of like powers of p , we obtain the following linear equations. The zeroth-order problem is given by Eq.(3), the first-order problem is given in the following form

$$L(u_1(x)) + g(x) = C_1 N_0(u_0(x)), \quad B\left(u_1, \frac{du_1}{dx}\right) = 0, \quad (6)$$

where $N_m(u_0(x), u_1(x), \dots, u_m(x))$ is the coefficient of p^m in the expansion of $N(v(x, p))$ about the embedding parameter p .

$$N(v(x, p, C_i)) = N_0(u_0(x)) + \sum_{m=1}^{\infty} N_m(u_0(x), u_1(x), \dots, u_m(x)) p^m. \quad (7)$$

It has been observed that the convergence of the series (5) depends upon the auxiliary constants C_1, C_2, C_3, \dots . If it is convergent at $p = 1$, one has

$$v(x, C_i) = u_0(x) + \sum_{k=1}^{\infty} u_k(x, C_1, C_2, \dots, C_k). \tag{8}$$

The result of the m th-order approximation is given

$$\tilde{u}(x, C_1, C_2, C_3, \dots, C_m) = u_0(x) + \sum_{i=1}^m u_i(x, C_1, C_2, \dots, C_i). \tag{9}$$

Substituting (9) into (1) yields the following residual

$$R(x, C_1, C_2, C_3, \dots, C_m) = L(\tilde{u}(x, C_1, C_2, C_3, \dots, C_m)) + g(x) + N(\tilde{u}(x, C_1, C_2, C_3, \dots, C_m)). \tag{10}$$

If $R = 0$, then \tilde{u} will be the exact solution. Generally such a case will not arise for nonlinear problems, but we can minimize the functional

$$J(C_1, C_2, C_3, \dots, C_m) = \int_a^b R^2(x, C_1, C_2, C_3, \dots, C_m) dx, \tag{11}$$

where a and b are the endpoints of the given problem. The unknown constants C_i ($i = 1, 2, 3, \dots, m$) can identified from the conditions

$$\frac{\partial J}{\partial C_1} = \frac{\partial J}{\partial C_2} = \dots = \frac{\partial J}{\partial C_m} = 0. \tag{12}$$

With these constants known, the approximate solution (of order m) is well determined.

3 Numerical Examples

To illustrate the validity and capability of the presented procedure, we shall consider the following two examples of multipoint two-points BVPs.

3.1 Example 1

Consider the following third order linear multipoint BVP [21, 23].

$$u'''(x) - k^2 u'(x) + 1 = 0, \quad u'(0) = 0, \quad u'(1) = 0, \quad u(0.25) = 0. \tag{13}$$

The exact solution of this problem is given by

$$u(x) = \frac{1}{k^3} \left(\sinh\left(\frac{k}{2}\right) - \sinh(kx) \right) + \frac{1}{k^2} \left(x - \frac{1}{2} \right) + \frac{1}{k^3} \left(\cosh(kx) - \cosh\left(\frac{k}{2}\right) \right) \tanh\left(\frac{k}{2}\right) \tag{14}$$

According to the OHAM formulation described in above section, we construct a homotopy equation in case of the physical constant $k = 5$ in the form of

$$(1-p) \left[\frac{\partial^3 v(x;p)}{\partial x^3} + 1 \right] = H(p) \left[\frac{\partial^3 v(x;p)}{\partial x^3} - 25 \frac{\partial v(x;p)}{\partial x} + 1 \right],$$

$$B \left(v(x,p), \frac{\partial v(x,p)}{\partial x} \right) = 0. \tag{15}$$

Now using Eq.(15) when $p = 0$, it yields the zeroth-order problem as:

$$u_0''(x) + 1 = 0, \quad u_0'(0) = 0, \quad u_0'(1) = 0, \quad u_0(0.25) = 0. \tag{16}$$

which has the solution

$$u_0(x) = 0.166667(-x^3 + 1.5x^2 - 0.25). \quad (17)$$

Now, apply Eq.(6) to give the first-order problem as:

$$\begin{aligned} u_1^{(3)}(x) = & 12.5c_{10}x^{11} + 12.5c_9x^{10} - 12.5c_{10}x^{10} + 12.5c_8x^9 - 12.5c_9x^9 + 12.5c_7x^8 \\ & -12.5c_8x^8 + 12.5c_6x^7 - 12.5c_7x^7 + 12.5c_5x^6 - 12.5c_6x^6 + 12.5c_4x^5 \\ & -12.5c_5x^5 + 12.5c_3x^4 - 12.5c_4x^4 + 12.5c_2x^3 - 12.5c_3x^3 + 12.5c_1x^2 \\ & -12.5c_2x^2 - 12.5c_1x + 0, \quad u'(0) = 0, \quad u'(1) = 0, \quad u(0.25) = 0. \end{aligned} \quad (18)$$

Substituting the solution of Eq. (18) together with Eq.(17) into Eq. (9) with $m = 1$, yields, the first order-approximate solution in the following form

$$\begin{aligned} \tilde{u}(x, C_1, \dots, C_{10}) = & 0.00572344c_{10}x^{14} + 0.00728438c_9x^{13} - 0.00728438c_{10}x^{13} \\ & +0.0094697c_8x^{12} - 0.0094697c_9x^{12} + 0.0126263c_7x^{11} \\ & -0.0126263c_8x^{11} + 0.0173611c_6x^{10} - 0.0173611c_7x^{10} \\ & +0.0248016c_5x^9 - 0.0248016c_6x^9 + 0.0372024c_4x^8 \\ & -0.0372024c_5x^8 + 0.0595238c_3x^7 - 0.0595238c_4x^7 \\ & -0.104167c_3x^6 + 0.104167c_3x^2 + 0.0595238c_4x^2 \\ & +0.0372024c_5x^2 + 0.0248016c_6x^2 + 0.0173611c_7x^2 \\ & +0.0126263c_8x^2 + 0.0094697c_9x^2 + 0.00728438c_{10}x^2 \\ & +c_2(0.104167x^6 - 0.208333x^5 + 0.208333x^2 - 0.0472005) \\ & +c_1(0.208333x^5 - 0.520833x^4 + 0.520833x^2 - 0.104167) \\ & -0.0248791c_3 - 0.0145612c_4 - 0.00920371c_5 - 0.00616891c_6 \\ & -0.00432949c_7 - 0.00315271c_8 - 0.002366c_9 - 0.00182056c_{10} \\ & +0.166667(-1.x^3 + 1.5x^2 - 0.25) \end{aligned} \quad (19)$$

Following the procedure described in section 2 on the domain between $a = 0$ and $b = 1$, using the residual error,

$$R = \tilde{u}'''(x, C_1, \dots, C_{10}) - 25\tilde{u}'(x, C_1, \dots, C_{10}) + 1. \quad (20)$$

The least-square method can be applied as

$$J(C_1, C_2, \dots, C_{10}) = \int_0^1 R^2 dx \quad (21)$$

and

$$\frac{dJ}{dC_1} = \frac{dJ}{dC_2} = \dots = \frac{dJ}{dC_{10}}.$$

Thus, the following optimal values of C_i 's are obtained:

$$\begin{aligned} C_1 = -0.394646, \quad C_2 = 0.605372, \quad C_3 = -1.03921, \\ C_4 = 1.04537, \quad C_5 = -1.01292, \quad C_6 = 0.720373, \\ C_7 = -0.468559, \quad C_8 = 0.209234, \quad C_9 = -0.0650757, \\ C_{10} = 0.00541865. \end{aligned}$$

By considering these values our approximate solution becomes,

$$\begin{aligned} \tilde{u}(x, C_1, \dots, C_{10}) = & -0.0121071 + 0.0986614x^2 - 0.166667x^3 + 0.205545x^4 \\ & -0.208337x^5 + 0.17131x^6 - 0.124082x^7 + 0.0765735x^8 \\ & -0.0429885x^9 + 0.0206412x^{10} - 0.00855799x^{11} \\ & +0.00259763x^{12} - 0.000513508x^{13} + 0.0000310133x^{14} \end{aligned} \quad (22)$$

Table 1: Comparison of exact solution and OHAM solution for Example 1 in case of $k = 5$

x	Exact Solution	OHAM Solution	Absolute Error	Absolute Error [21]	Absolute Error [23]
0.0	-0.0121071	-0.0121071	12.19×10^{-12}	1.3×10^{-10}	6.65×10^{-5}
0.2	-0.0092222	-0.0092222	1.66×10^{-12}	6.99×10^{-9}	5.25×10^{-5}
0.4	-0.0033202	-0.0033202	24.57×10^{-12}	1.07×10^{-8}	1.88×10^{-5}
0.60	0.0033202	0.0033202	2.44×10^{-12}	1.07×10^{-8}	1.73×10^{-5}
0.80	0.0092222	0.0092222	7.96×10^{-13}	4.98×10^{-5}	4.98×10^{-5}
1.00	0.0121071	0.0121071	2.18×10^{-12}	6.35×10^{-5}	6.35×10^{-5}

Table 2: Comparison of exact solution and OHAM solution for Example 1 in case of $k = 10$

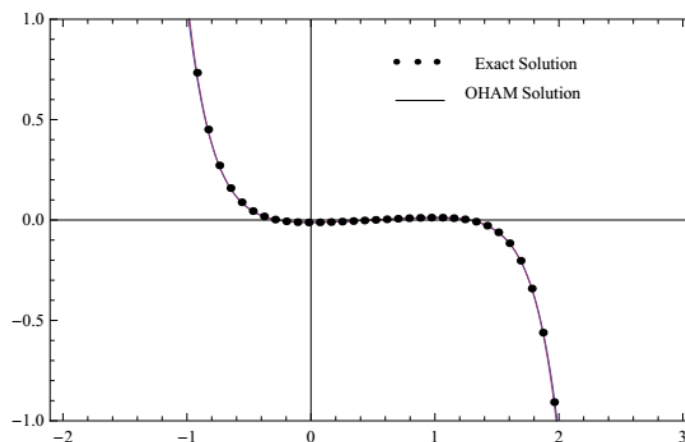
x	Exact Solution	OHAM Solution	Absolute Error	Absolute Error [23]
0.0	-0.00400009	-0.00400009	3.61×10^{-11}	3.52×10^{-5}
0.2	-0.00286501	-0.00286501	1.88×10^{-10}	3.03×10^{-5}
0.4	-0.000984164	-0.000984164	2.10×10^{-10}	1.40×10^{-5}
0.60	0.000984164	0.000984164	9.28×10^{-11}	7.0×10^{-6}
0.80	0.00286501	0.00286501	1.49×10^{-10}	1.96×10^{-5}
1.00	0.00400009	0.00400009	3.79×10^{-11}	2.40×10^{-5}

When the physical constant $k = 10$ and following the same procedure as previously applied in case of $k = 5$, we obtain the following first-order approximate solution

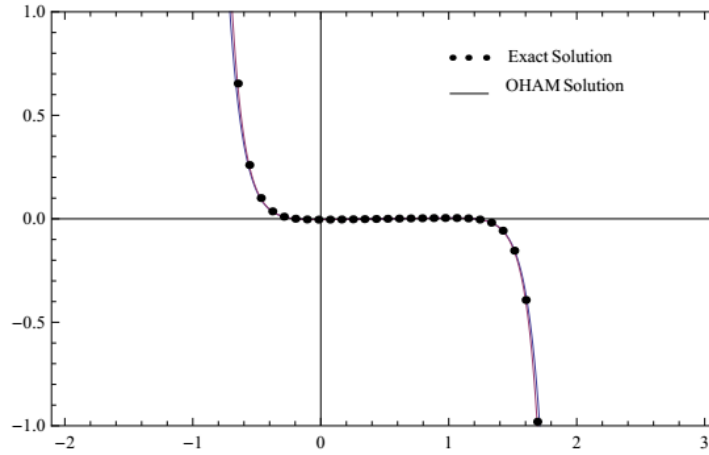
$$\begin{aligned}
 \tilde{u}(x, C_1, \dots, C_{10}) = & 0.0000310133x^{14} - 0.000513508x^{13} + 0.00259763x^{12} \\
 & - 0.00855799x^{11} + 0.0206412x^{10} - 0.0429885x^9 \\
 & + 0.0765735x^8 - 0.124082x^7 + 0.17131x^6 - 0.208337x^5 \\
 & + 0.205545x^4 - 0.166667x^3 + 0.0986614x^2 - 0.0121071
 \end{aligned} \tag{23}$$

Tables 1 and 2 show the comparison between the present solution obtained by using first-order OHAM approximation and the numerical results obtained from higher-order of approximation using other methods including OHAM solutions of three-order of approximations. Fig1, represents the plots of the first-order OHAM approximation the the exact one. Runge Kutta method

(a)



(b)



3.2 Example 2

Consider the following linear three-point non-local BVP [6],

$$u''''(x) - e^x u'''(x) + u(x) = 1 - e^x \cosh x + 2 \sinh x, \tag{24}$$

subject to the boundary conditions

$$\begin{aligned} u(0.25) &= 1 + \sinh(0.25), & u'(0.25) &= \cosh(0.25), \\ u''(0.25) &= \sinh(0.25), & u(0.5) - u(0.75) &= \sinh(0.5) - \sinh(0.75) \end{aligned}$$

Using the method presented in Section 2, we obtain the first-order approximate solutions in the following form:

$$\begin{aligned} \tilde{u}(x, C_1, C_2, C_3) &= -0.0003381x^9 + 0.00048716x^8 - 0.00710433x^7 + 0.00403864x^6 \\ &+ 0.00673227x^5 - 0.0276036x^4 - 6.85065x^3 - 5.68059e^{-x}x^2 \\ &+ 11.8151e^x x^2 - 0.177518e^{2x}x^2 - 31.4x^2 - 41.1715e^{-x}x \\ &- 103.409e^x x + 0.843612e^{2x}x - 225.325x - 97.8443e^{-x} \\ &+ 274.61e^x - 1.19609e^{2x} - 174.569 \end{aligned} \tag{25}$$

Table 3: comparison of exact solution and OHAM solution for Example 1.

x	Exact Solution	OHAM Solution	OHAM Error	Absolute Error	Absolute Error [5]
0.01	0.000000	0.999945	5.41×10^{-5}	1.02×10^{-4}	2.49×10^{-6}
0.21	1.20134	1.20134	4.31×10^{-7}	5.33×10^{-7}	1.95×10^{-8}
0.41	1.41075	1.41076	9.38×10^{-6}	7.60×10^{-6}	4.82×10^{-6}
0.61	1.63665	1.63672	6.5×10^{-5}	3.90×10^{-5}	3.94×10^{-6}
0.81	1.88811	1.88808	2.94×10^{-5}	2.42×10^{-6}	3.34×10^{-6}
1.02	1.1752	2.17428	9.25×10^{-4}	3.05×10^{-4}	9.63×10^{-5}

The obtained results are reported in Table 3. The performance of this procedure is very good and the result obtained during one-order of approximation is in a very good agreement to the exact solution comparing with other methods which needs higher order of approximation. This performance can be easily observed from this Table. The comparison of the exact solution and the approximate solution are shown in Figure 2.

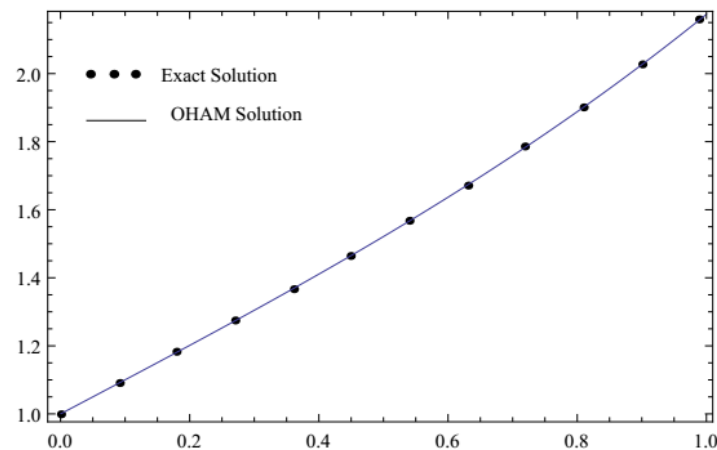


Figure 2: Exact and approximate solution using OHAM for Example 2.

4 Conclusions

In this research study, we proposed a new accurate approximate analytical solution for multipoint BVPs based on a simple modification of the optimal homotopy asymptotic method (OHAM) and comparing it with the results obtained by the same method previously and other method in literature. The examples presented in this work leads to the conclusion that the obtained results are quit accurate and are in a very good agreement with the analytical solution which is demonstrate and prove that this procedure is explicit, effective and accurate for this type of ordinary differential equations and can be easily applied to other differential equations.

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