Available online at www.refaad.com
Gen.Lett. Math., 8(2) (2020), 67-74
Research Article

General Letters in Mathematics (GLM)

Journal Homepage: http://www.refaad.com/views/GLM/home.aspx
ISSN: 2519-9277 (Online) 2519-9269 (Print)

Generalized the Divisor Sum T_k-Function of Graph

Hariwan Fadhil M.Salih[a,*], Nechirvan Badal Ibrahim [b,*]

[a]Department of Mathematics, College of Science, University of Duhok, Duhok-IRAQ.
[b]Department of Mathematics, College of Science, University of Duhok, Duhok-IRAQ.

Abstract

Let $G = (V, E)$ be a simple connected undirected graph. In this paper, we define generalized the divisor sum T_k-function of a graph which is the summation of the sum of divisor σ_k-function of the degree of the vertices of a graph denoted by $T_k(G) = \sum_{v \in V(G)} \sigma_k(deg(v))$ and determine the divisor sum T_k-function of some standard graphs and finally some results are proved.

Keywords: Generalized the divisor sum T_k-function, the divisor sum σ_k-function, More.
2010 MSC: 11A25, 05C99.

1. Introduction

The concepts of Number Theory were used starting from 1980 in Graph theory and many types of graphs have been studied. This paper extends results given in the work [8], in 2019, by M.Salih and Ibrahim which defines the generalized Euler's Φ-Function of a graph which is the summation of the Euler's ϕ-function of the degree of the vertices of a graph and it is denoted by $\Phi(G)$. There are variety of graphs defined through arithmetic conditions where unitary Cayley graphs is an important example which can be found in [13]. There are also many interesting connections, in [13], between number theory and many other mathematical topics. The notion of divisor graphs $G_{((D(n)))}$ was introduced by Singh and Santhosh [9] and in [11] Kannan, Narasimhan and Shanmugavelan the divisor function graph was introduced and studied its properties, whereas, the concept of the Euler function graph $G(\phi(n))$ was studied in [14], in 2017, by Shanmugavelan and some graph theoretical properties of two derivative Euler Phi function set-graphs was studied in [12], in 2019, by Kok, Mphako-Banda and Naduvath. For all other standard terminologies and notations we follow [[3], [6], [7], [10], [5], [4], [15]].
The divisor sum σ_k-function in number theory was first studied by Ramanujan, which is a number of crucial congruences and identities were given by him; these are preserved separately in [1]. The sum of the divisor σ_k-function is a related function to the divisor function, denoted by $\sigma_k(n)$, is counting the sum of positive divisors of n to the power k was studied in [2]. For instance, 8 has 4 positive divisors, then $\sigma_2(8) = 85$. Also, there are 2 positive divisors of 11, then $\sigma_2(11) = 122$. The above example shows that if $(n = p)$ is prime, then $\sigma_k(p) = p^k + 1$ and in general, $\sigma_k(p^a) = \frac{p^{(a+1)k}-1}{p^k-1}$ for any positive integer

*Corresponding author
Email addresses: hariwan.msalih@uod.ac (Hariwan Fadhil M.Salih), nechirvan.badal@uod.ac (Nechirvan Badal Ibrahim)

doi:https://doi.org/10.31559/glm2020.8.2.4

Received 25 Nov 2019 : Accepted 18 May 2020

a. For any positive integer n we have that $\sigma_k(n) = \sum_{d|n} d^k = \prod_{i=1}^{t} \frac{p^{(a_i+1)k}-1}{p_i^k-1}$ where $n = p_1^{a_1} p_2^{a_2} \dots p_t^{a_t}$. Notice that when $k = 0$, we say that $\sigma_0(n)$ is the number of divisors which is also denoted by $\tau(n)$ and when $k = 1$, we say that $\sigma_1(n)$ is the sum of the divisors of n which is also denoted by $\sigma(n)$ [[3], [6], [7]]. The Euler $\phi-$function is counted as one of the typical area in number theory defined as the number of positive integers less than n which are relatively prime (or co-prime) to n [3]. For instance, there are 4 positive integers less than 10 which are relatively prime to 10.

A function f is said to be multiplicative if for all positive integers m, nsuch that m, n are relatively prime, then $f(mn) = f(m)f(n)$. The sum of the divisor σ_k-function is multiplicative [2].

In this paper, we attempt to use a number theory function called the divisor σ_k-function into graph theory and we define the divisor sum T_k-function T_k (G) of the graph G, which is counting the sum of the sum of the positive divisor σ_k-function for the degree of vertices of a graph G. It is shown a relationship between the Euler's ϕ-function and generalized the divisor sum T_k, where $k = 1$.

2. Generalized The Divisor Sum T_k-Function of Some Standard Graphs

In this section, we determine generalized the divisor sum T_k-function of some important graphs in graph theory which are the path graph P_n, cycle graph C_n, complete graph K_n, complete bipartite graph $K_{m,n}$, k-partite graph $K_{m_1,m_2,\dots,m_n}$, star graph S_n, and wheel graph W_n.

Definition 2.1. Let G be a simple connected graph and let $\sigma_k(deg(v))$ be defined as the sum of the divisor σ_k-function of the degree of vertices v of a graph G which is denoted by $T_k(G)$. Then

$$T_k(G) = \sum_{v \in V(G)} \sigma_k(deg(v)).$$

In the following proposition, we determine the general form and exact values of generalized the divisor sum T_k-function of some standard graphs.

Proposition 2.2.

1. *Generalized the divisor sum T_k-function of the path graph $G = P_n$, for $n \geqslant 2$ vertices, is* $T_k(P_n) = (2^k + 1)(n - 2) + 2$.
2. *Generalized the divisor sum T_k-function of the cycle graph $G = C_n$, for $n \geqslant 3$ vertices, is* $T_k(C_n) = (2^k + 1)n$
3. *Generalized the divisor sum T_k-function of the complete graph $G = K_n$, for $n \geqslant 3$ vertices, is*

$$T_k(K_n) = n * \sigma_k(n-1) = n * \left(\prod_{i=1}^{t} \frac{p_i^{(a_i+1)k} - 1}{p_i^k - 1} \right)$$

   where $(n - 1) = p_1^{a_1} p_2^{a_2} \dots p_t^{a_t}$.
4. *Generalized the divisor sum T_k-function of the complete bipartite graph $G = K_{m,n}$, for any positive integers m, n vertices, is*

$$T_k(K_{m,n}) = (m * \sigma_k(n)) + (n * \sigma_k(m)) = \left(m * \prod_{i=1}^{t} \frac{p_i^{(a_i+1)k} - 1}{p_i^k - 1} \right) + \left(n * \prod_{j=1}^{r} \frac{p_j^{(a_j+1)k} - 1}{p_j^k - 1} \right)$$

5. *Generalized the divisor sum T_k-function of the star graph $G = S_n$, for $n \geqslant 2$ vertices, is*

$$T_k(S_n) = \sigma_k(n-1) + (n-1) = \left(\prod_{i=1}^{t} \frac{p_i^{(a_i+1)k} - 1}{p_i^k - 1} \right) + (n-1)$$

   where $(n - 1) = p_1^{a_1} p_2^{a_2} \dots p_t^{a_t}$.

6. *Generalized the divisor sum* T_k*-function of the* k*-partite graph* $G = K_{m_1,m_2,\dots,m_n}$*, for any positive integers* $m_1, m_2, \dots, m_n$ *vertices, is*

$$\begin{aligned} T_k(K_{m_1,m_2,\dots,m_n}) &= (m_1 * \sigma_k(m_2 + m_3 + \cdots + m_n)) + (m_2 * \sigma_k(m_1 + m_3 + \cdots + m_n)) \\ &+ (m_n * \sigma_k(m_1 + m_2 + \cdots + m_{n-1})) \\ &= \sum_{i=1}^{n} \left(m_i * \sigma_k \left(\sum_{j \neq i, j=1,2,\dots,n} m_j \right) \right) = \sum_{i=1}^{n} \left(m_i * \left(\prod_{j \neq i, j=1,2,\dots,n} \frac{p_i^{(a_i+1)k} - 1}{p_i^k - 1} \right) \right) \end{aligned}$$

where

$$\left(\sum_{j=1,2,\dots,n} m_j = p_1^{a_1} p_2^{a_2} \dots p_t^{a_t} \right)$$

7. *Generalized the divisor sum* T_k*-function of the Wheel graph* $G = W_n$*, for* $n \geqslant 4$ *vertices, is*

$$T_k(W_n) = \sigma_k(n-1) + (3^k + 1)(n-1) = \prod_{i=1}^{t} \frac{p_i^{(a_i+1)k} - 1}{p_i^k - 1} + (3^k + 1)(n-1)$$

where $(n-1) = p_1^{a_1} p_2^{a_2} \dots p_t^{a_t}$.

Proof. 1. The path graph Pn of order n, if $n = 2$, we have two vertices of degree one, then we have $T_k(P_2) = \sigma_k(1) + \sigma_k(1) = 2$. If $n \geqslant 3$, we have two vertices of degree one and $n-2$ vertices of degree two, then we have:

$$\begin{aligned} T_k(P_3) &= \sigma_k(2) + 2\sigma_k(1) = (2^k + 1) + 2; \\ T_k(P_4) &= 2\sigma_k(2) + 2\sigma_k(1) = 2(2^k + 1) + 2; \\ T_k(P_5) &= 3\sigma_k(2) + 2\sigma_k(1) = 3(2^k + 1) + 2; \\ &\vdots \\ T_k(P_n) &= (n-2)\sigma_k(2) + 2\sigma_k(1) = (n-2)(2^k + 1) + 2. \end{aligned}$$

2. In a cycle graph Cn of order n, we have n vertices of degree two, and then we have:

$$\begin{aligned} T_k(C_n) &= \sum_{v \in V(C_n)} \sigma_k(deg(v)) = \sigma_k(deg(v_1)) + \sigma_k(deg(v_2)) + \cdots + \sigma_k(deg(v_n)) \\ &= n * \sigma_k(2) = n(2^k + 1). \end{aligned}$$

3. In a complete graph K_n of order n, we have n vertices of degree $n-1$, and then we have:

$$\begin{aligned} T_k(K_n) &= \sum_{v \in V(K_n)} \sigma_k(deg(v)) = \sigma_k(deg(v_1)) + \sigma_k(deg(v_2)) + \cdots + \sigma_k(deg(v_n)) \\ &= n * \sigma_k(n-1) = n * \left(\prod_{i=1}^{t} \frac{p_i^{(a_i+1)k} - 1}{p_i^k - 1} \right). \end{aligned}$$

where $(n-1) = p_1^{a_1} p_2^{a_2} \dots p_t^{a_t}$.

4. In a complete bipartite graph $K_{m,n}$ of order $m+n$, we have m vertices of degree n and n vertices of degree m, then we have:

$$\begin{aligned} T_k(K_{m,n}) = \sum_{v \in V(K_{m,n})} \sigma_k(deg(v)) &= \sigma_k(deg(u_1)) + \sigma_k(deg(u_2)) + \cdots + \sigma_k(deg(u_m)) \\ &+ \sigma_k(deg(v_1)) + \sigma_k(deg(v_2)) + \cdots + \sigma_k(deg(v_n)) \\ &= \sigma_k(n) + \sigma_k(n) + \cdots + \sigma_k(n) + \sigma_k(m) + \sigma_k(m) + \cdots + \sigma_k(m) \\ &= (m * \sigma_k(n)) + (n * \sigma_k(m)) \\ &= \left(m * \prod_{i=1}^{t} \frac{p_i^{(a_i+1)k} - 1}{p_i^k - 1} \right) + \left(n * \prod_{j=1}^{r} \frac{p_j^{(a_j+1)k} - 1}{p_j^k - 1} \right). \end{aligned}$$

5. In a star graph S_n of order n, we have $n-1$ vertices of degree one and we have one vertex of degree $n-1$, say v_1, then we have:

$$\begin{aligned} T_k(S_n) &= \sum_{v \in V(S_n)} \sigma_k(deg(v)) = \sigma_k(deg(v_1)) + \sigma_k(deg(v_2)) + \cdots + \sigma_k(deg(v_n)) \\ &= \sigma_k(n-1) + (n-1) = \prod_{i=1}^{t} \frac{p_i^{(a_i+1)k} - 1}{p_i^k - 1} + (n-1). \end{aligned}$$

where $(n-1) = p_1^{a_1} p_2^{a_2} \ldots p_t^{a_t}$.

6. In a complete k-partite graph $K_{(m_1,m_2,\ldots,m_n)}$ of order $m_1 + m_2 + \cdots + m_n$, we have m_i vertices of degree $\sum_{j \neq i} m_j$ where $j, i = 1, 2, \ldots n$, and then we have:

$$\begin{aligned} T_k(K_{m_1,m_2,\ldots,m_n}) &= \sum_{v \in V(K_{m_1,m_2,\ldots,m_n})} \sigma_k(deg(v)) = \sigma_k(deg(v_1)) + \sigma_k(deg(v_2)) + \cdots + \sigma_k(deg(v_n)) \\ &= (m_1 * \sigma_k(m_2 + m_3 + \cdots + m_n)) + (m_2 * \sigma_k(m_1 + m_3 + \cdots + m_n)) \\ &+ (m_n * \sigma_k(m_1 + m_2 + \cdots + m_{n-1})) \\ &= \sum_{i=1}^{n} \left(m_i * \sigma_k \left(\sum_{j \neq i, j=1,2,\ldots,n} m_j \right) \right) = \sum_{i=1}^{n} \left(m_i * \left(\prod_{j \neq i, j=1,2,\ldots,n} \frac{p_i^{(a_i+1)k} - 1}{p_i^k - 1} \right) \right). \end{aligned}$$

where

$$\left(\sum_{j=1,2,\ldots,n} m_j = p_1^{a_1} p_2^{a_2} \ldots p_t^{a_t} \right)$$

7. In a wheel graph Wn of order n, we have $n-1$ vertices of degree three and we have one vertex of degree $n-1$, say v_1, then we have:

$$\begin{aligned} T_k(W_n) &= \sum_{v \in V(W_n)} \sigma_k(deg(v)) = \sigma_k(deg(v_1)) + \sigma_k(deg(v_2)) + \cdots + \sigma_k(deg(v_n)) \\ &= \sigma_k(n-1) + (3^k + 1)(n-1) = \prod_{i=1}^{t} \frac{p_i^{(a_i+1)k} - 1}{p_i^k - 1} + (3^k + 1)(n-1). \end{aligned}$$

where $(n-1) = p_1^{a_1} p_2^{a_2} \ldots p_t^{a_t}$.

□

3. Some Results and Their Proofs

In this section, we give some new results of finding the generalized the divisor T_k-function of a graph and we also give a relationship between the Euler's ϕ-function and generalized the divisor sum T_k- function, where $k = 1$. The following theorem is the most useful properties of generalized the divisor T_k-function for the divisor sum of the degree of the vertices in a graph.

Theorem 3.1. *For all $v \in V(G)$, $\deg(v) = p_1^{a_1} p_2^{a_2} \dots p_t^{a_t} \geqslant 1$, we have*

$$T_k(G) = \sum_{v \in V(G)} \sigma_k(\deg(v)) = \sum_{v \in V(G)} \left(\prod_{p_i \mid \deg(v)} \left(\sum_{j=0}^{a_i} p_i^{jk} \right) \right) = \sum_{v \in V(G)} \left(\prod_{i=1, p_i \mid \deg(v)}^{t} \frac{p_i^{(a_i+1)k} - 1}{p_i^k - 1} \right)$$

Proof. Let $v \in V(G)$, $(\deg(v) = p_1^{a_1} p_2^{a_2} \dots p_t^{a_t}) \geqslant 1$ and since σ_k is multiplicative, thus

$$\begin{aligned} T_k(G) &= \sum_{v \in V(G)} \sigma_k(\deg(v)) = \sum_{v \in V(G)} \sigma_k(p_1^{a_1} p_2^{a_2} \dots p_t^{a_t}) \\ &= \sum_{v \in V(G)} \left(\sigma_k(p_1^{a_1}) * \sigma_k(p_2^{a_2}) * \dots * \sigma_k(p_t^{a_t}) \right) \\ &= \sum_{v \in V(G)} \left(\left(\frac{p_1^{(a_1+1)k} - 1}{p_1^k - 1} \right) * \left(\frac{p_2^{(a_2+1)k} - 1}{p_2^k - 1} \right) * \dots * \left(\frac{p_t^{(a_t+1)k} - 1}{p_t^k - 1} \right) \right) \\ &= \sum_{v \in V(G)} \left(\prod_{i=1, p_i \mid \deg(v)}^{t} \frac{p_i^{(a_i+1)k} - 1}{p_i^k - 1} \right) \end{aligned}$$

Or we can use the geometric formula for series and the result will follow. □

Lemma 3.2. *If $\deg(v) = p^a \geqslant 1$ for $v \in V(G)$, we have*

$$T_k(G) = \sum_{d \mid p^a} \sigma_k(p^a) = \begin{cases} \sum_{d \mid p^a} \frac{p^{(a+1)k} - 1}{p^k - 1} & \text{if } k \neq 0 \\ \sum_{d \mid p^a} (a+1) & \text{if } k = 0 \end{cases}$$

Proof. Follows from the Theorem 3.1. □

Theorem 3.3. *For all $v \in V(G)$, $\deg(v) \geqslant 1$, we have*

$$T_k(G) = \sum_{v \in V(G)} \left(\sum_{\deg(v)=1}^{N} \sigma_k(\deg(v)) \right) = \sum_{v \in V(G)} \left(\sum_{\deg(v)=1}^{N} \left((\deg(v))^k \left[\frac{N}{\deg(v)} \right] \right) \right)$$

where $\left[\frac{N}{\deg(v)} \right]$ is the greatest integer less than $\left(\frac{N}{\deg(v)} \right)$ and N is a positive integer.

Proof. Set, for all $v \in V(G)$, $B_{(N, \deg(v))} = \left[\frac{N}{\deg(v)} \right] - \left[\frac{N-1}{\deg(v)} \right]$ for $\deg(v) \geqslant 1$. Then

$$(\deg(v))^k (B_{(N, \deg(v))}) = \begin{cases} (\deg(v))^k & \text{when } \deg(v) \mid N \\ 0 & \text{when } \deg(v) \nmid N \end{cases}$$

and then

$$\begin{aligned}
T_k(G) &= \sum_{v\in V(G)}\left(\sum_{deg(v)=1}^{N}\sigma_k(deg(v))\right) = \sum_{v\in V(G)}\left(\sum_{deg(v)=1}^{N}\left(\sum_{k=1}^{deg(v)}k^{deg(v)}B_{(deg(v),k)}\right)\right)\\
&= \sum_{v\in V(G)}\left(\sum_{deg(v)=1}^{N}\left(\sum_{k=1}^{deg(v)}k^{deg(v)}\left(\left[\frac{deg(v)}{k}\right]-\left[\frac{deg(v)-1}{k}\right]\right)\right)\right)\\
&= \sum_{v\in V(G)}\left(\sum_{deg(v),k=1}^{N}k^{deg(v)}\left[\frac{deg(v)}{k}\right]-\sum_{1\leqslant k\leqslant N,\ 1\leqslant deg(v)\leqslant N-1}k^{deg(v)}\left[\frac{deg(v)}{k}\right]\right)\\
&= \sum_{v\in V(G)}\sum_{1\leqslant k\leqslant N,\ deg(v)=N}k^{deg(v)}\left[\frac{deg(v)}{k}\right]
\end{aligned} \tag{3.1}$$

The results follos from the equation (3.1) by exchanging k with $deg(v)$ and since $deg(v) = N$, we obtain that

$$T_k(G) = \sum_{v\in V(G)}\left(\sum_{deg(v)=1}^{N}\sigma_k(deg(v))\right) = \sum_{v\in V(G)}\left(\sum_{deg(v)=1}^{N}\left((deg(v))^k\left[\frac{N}{deg(v)}\right]\right)\right)$$

□

Lemma 3.4. *For all $v \in V(G)$, we have that $deg(v)$ is a prime number if and only if*

$$T_k(G) = \sum_{\forall v\in V(G)}\sigma_k(deg(v)) = \sum_{\forall v\in V(G)}\left((deg(v))^k+1\right).$$

Proof. If $deg(v)$ is a prime $\forall v \in V(G)$, then if we have t-vertices in a graph G.

$$T_k(G) = \sum_{\forall v\in V(G)}\sigma_k(deg(v)) = \sigma_k(deg(v_1)) + \sigma_k(deg(v_2)) + \cdots + \sigma_k(deg(v_t))$$

Since $\forall v \in V(G)$, $deg(v)$ is a prime number. Therefore $deg(v_1) = deg(v_2) = \cdots = deg(v_t) = p$

$$\begin{aligned}
T_k(G) &= \sum_{\forall v\in V(G)}\sigma_k(deg(v)) = \sigma_k(p)+\sigma_k(p)+\cdots+\sigma_k(p) = \sum_p \sigma_k(p)\\
&= \sum_p (p^k+1) = \sum_{\forall v\in V(G)}\left((deg(v))^k+1\right).
\end{aligned}$$

□

The following theorem gives a relationship between the Euler's ϕ-function and generalized the divisor sum T_k-function of a graph, where $k = 1$.

Theorem 3.5. *For all $v \in V(G)$, $deg(v) \geqslant 1$, we have that*

$$T_1(G) = \sum_{v\in V(G)}\sigma_1(deg(v)) = \sum_{v\in V(G)}\left(\sum_{d|deg(v)}\phi(d)*\sigma_0\left(\frac{deg(v)}{d}\right)\right),$$

where d is the divisor of the degree of vertices in a graph G.

Proof. Let $(\deg(v)) = p_1^{a_1}p_2^{a_2}\dots p_t^{a_t}$. If we choose the left hand side (LHS) to be $\sum_{v\in V(G)}\sigma_1(\deg(v))$. Since the divisor function σ_k and the Euler's ϕ-function are multiplicative and from the Theorem 3.1, we obtain in the case when $k=1$, that

$$\sum_{v\in V(G)}\sigma_1(\deg(v)) = \sum_{\deg(v)=\{p_1,p_2,\dots,p_t\}}\sigma_1(p_1^{a_1}p_2^{a_2}\dots p_t^{a_t}) = \sum_{\deg(v)=\{p_1,p_2,\dots,p_t\}}\left(\prod_{i=1}^{t}\frac{p_i^{(a_i+1)k}-1}{p_i^k-1}\right)$$

This side will be called the left hand side (LHS). If we see the other side the right hand side (RHS). In our case d is our divisor of $\deg(v)$ which has the form $d = p_1^{i_1}p_2^{i_2}\dots p_t^{i_t}$ where $0\leqslant i_1,i_2,\dots,i_t\leqslant a_1,a_2,\dots,a_t$ respectively. The representation of each of the divisors will be allowed because in this way the divisors with different primes can be counted with different these primes could have. Since $\deg(v) = p_1^{a_1}p_2^{a_2}\dots p_t^{a_t}$. Thus,

$$\sum_{v\in V(G)}\left(\sum_{d|\deg(v)}\phi(d)*\sigma_0\left(\frac{\deg(v)}{d}\right)\right) = \sum_{\{p_1,p_2,\dots,p_t\}}\left(\sum_{i_1=0}^{a_1}\sum_{i_2=0}^{a_2}\cdots\sum_{i_t=0}^{a_t}\phi(p_1^{i_1}p_2^{i_2}\dots p_t^{i_t})*\sigma_0\left(\frac{p_1^{a_1}p_2^{a_2}\dots p_t^{a_t}}{p_1^{i_1}p_2^{i_2}\dots p_t^{i_t}}\right)\right)$$

Since ϕ and σ_0 are multiplicative and by rearranging their terms, we obtain

$$\text{RHS} = \sum_{\{p_1,p_2,\dots,p_t\}}\left(\sum_{i_1=0}^{a_1}\sum_{i_2=0}^{a_2}\cdots\sum_{i_t=0}^{a_t}\phi(p_1^{i_1})\sigma_0(p_1^{a_1-i_1})*\phi(p_2^{i_2})\sigma_0(p_2^{a_2-i_2})*\dots*\phi(p_t^{i_t})\sigma_0(p_t^{a_t-i_t})\right)$$

Since we have t-terms and for each of them we only have two terms to be variable and the rest are constant which can be obtained in front of the sum, so by applying this way t-times we obtain that

$$\text{RHS} = \sum_{\{p_1,p_2,\dots,p_t\}}\left(\sum_{i_1=0}^{a_1}\phi(p_1^{i_1})\sigma_0(p_1^{a_1-i_1})*\sum_{i_2=0}^{a_2}\phi(p_2^{i_2})\sigma_0(p_2^{a_2-i_2})*\dots*\sum_{i_t=0}^{a_t}\phi(p_t^{i_t})\sigma_0(p_t^{a_t-i_t})\right) \tag{3.2}$$

If the first sum in the bracket is denoted by A, then

$$\begin{aligned} A &= \sum_{i_1=0}^{a_1}\phi(p_1^{i_1})\sigma_0(p_1^{a_1-i_1}) \\ &= \phi(1)\sigma_0(p_1^{a_1}) + \phi(p_1)\sigma_0(p_1^{a_1-1}) + \dots + \phi(p_1^{i_1-1})\sigma_0(p_1) + \phi(p_1^{a_1})\sigma_0(1) \\ &= 1 + p_1 + p_1^2 + p_1^3 + \dots + p_1^{a_1-1} + p_1^{a_1} \end{aligned}$$

It can be seen that this is a geometric progression, so we can use its formula here,

$$A = \sum_{i_1=0}^{a_1}\phi(p_1^{i_1})\sigma_0(p_1^{a_1-i_1}) = \frac{1-p_1^{a_1+1}}{1-p_1} = \frac{p_1^{a_1+1}-1}{p_1-1}$$

This result can be applied to all the t-terms in (3.2), so the RHS will be

$$\text{RHS} = \sum_{\{p_1,p_2,\dots,p_t\}}\left(\frac{p_1^{a_1+1}-1}{p_1-1}*\frac{p_2^{a_2+1}-1}{p_2-1}*\dots*\frac{p_t^{a_t+1}-1}{p_t-1}\right) = \sum_{\{p_1,p_2,\dots,p_t\}}\left(\prod_{i=1}^{t}\frac{p_i^{a_i+1}-1}{p_i-1}\right) = \text{LHS}$$

□

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